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OF
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BY

HORACE WILMER MARSH

Head of Department of Mathematics, School of Science and Technology, Pratt Institute

VOLUME III

TECHNICAL GEOMETRY

> "These are the realities;
> We are the shadows."

FIRST EDITION

FIRST THOUSAND

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PREFACE

IN this period of education when every study is measured by its vocational value, one must be wise indeed not to be misled into the belief that geometry is of value in the business or industry or profession in which one shall engage, only when its study is interspersed with every conceivable application. Yet the supreme value of demonstrative geometry to a boy or girl or to an older person, consists not in the various ways in which it may be applied, but in the fact that its study, when uninterrupted by extraneous material, gives the ability to think clearly and logically.

The development of reason through the use of the mind in reasoning, and the mind's inherent self-activity, are therefore two of the unchallenged facts which underly the subject-matter and method of this book. The knowledge and acceptance of these principles are as general as is their disregard. With hardly an exception demonstrative geometries from Euclid to the present, in method of presentation are essentially Euclidean. Theorem, figure, hypothesis, conclusion, and " Q. E. D.", succeed each other in undeviating order. The results of the use of this method are inevitable. Students naturally mathematical acquire a formal knowledge of geometry and a restricted development of reason; others develop memory of demonstrations, only.

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Counteractive devises, such as inventive and mental geometry, and the requirement of many originals, although in general use, have not materially affected these results, In the study of the great body of theorems—drawn figures, stated hypotheses and conclusions, and printed demonstrations, make thought, origination, creation, and reason, unnecessary for all, impossible for many. To the large majority, knowledge of geometry has meant little more than a repetition of the words of another, and the possession of thought through *thinking* has been exceptional. The relation of the unknown to the known has not been recognized, the act of their relation having been performed by another. The increasing number of exercises, supplementary theorems, and foreign matter in the later geometries only serve to emphasize the fact that educators generally, realize that given figures and printed demonstrations are not suitable stimuli to reason, or factors in development, but that *direct contact with demonstrable truth is necessary.*

In this text this contact is effected in a unique but strictly scientific manner which may be most readily explained by reference to the first theorem. In the development of that theorem, a student locates a point, draws two lines in opposite directions from it, by a study of the definitions discovers that he has drawn a straight angle, by measurement with the protractor determines that it equals 180°, and with exactness and in inevitable form writes theorem 1 in answer to the question "Therefore what kind of angle equals how many degrees?"

A simple, natural process: construction, a name, investigation, generalization; a process which replaces the formal statement of unknown words illustrated by a picture and formulated in printed hypothesis and conclusion; nothing to be committed to memory, but only something to be done, a name to be sought, a measurement to be taken, a truth to be written. The conception thus formed and predicated, the student visualizes in a figure which to him is not definable as "*Any* combination of points, lines," etc., but is *his* linear representation of the thing, or relation of things specified in the theorem. Then follows an *hypothesis, conclusion, and demonstration of the student's own creation.* From the first statement to the final word of the demonstration, mental action is vigorous and comprehensive, limited only by the expressed conditions of the theorem itself, the objective becoming existent only through the creative expression of the student.

The demonstrations of the truths of geometry are not forced upon the mind from without, as burdensome to the memory and foreign to the student's normal, mental life, but are developed through individual, unrestricted, necessitated effort. They are *learned* not by a study of key demonstrations, but *by demonstrating.*

A wholesomeness of growth, a vigor and clearness of thought, and a harmony of mental development unattainable by the old, objective methods, are logical and inevitable. Geometry is originated and understood, and other subjects seem easier and more interesting. Self-realization through self-expression is therefore more than an ideal; it is a reality.

Attention is invited to the fact that only commensurable cases are demonstrated in the successive books of plane geometry, the theory of limits with all incommensurables being presented in the last book. By this arrangement the theory and its application, for the first time become intelligible and therefore worth while.

In a book without "pictures" it is necessary to define much more accurately than in one in which an incomplete or faulty definition is supplemented by a graphical representation. Every definition herein listed is strictly **logical**

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TECHNICAL GEOMETRY

Three additional axioms are developed in the text; two others precede the first book of solid geometry.

6. A Theorem. Like an axiom a **theorem** is a statement of a truth. This truth may be self-evident but is generally of such nature that the mind does not accept it without proof. Even when self-evident, a theorem is based on simpler, underlying truths, by means of which it may be demonstrated.

Therefore **a theorem is a demonstrable truth.**

When Book I is completed the student will have found that all of its theorems are self-evident and will no doubt question the necessity of their demonstration. The further study of geometry, however, will reveal the fact that without these fundamental truths it is impossible to demonstrate the theorems which follow.

7. A Demonstration. The **proof** of a theorem is the means by which its truth is established; it is called a **demonstration.**

A **demonstration** consists of a series of statements in which each successive statement is true only because of the truth of one or more preceding statements, on the basis of one or more accepted **mathematical authorities,** viz., an axiom, definition, specifications of the theorem (hypothesis), or a theorem previously demonstrated.

Therefore **a demonstration is a logically arranged series of statements,** *each based on mathematical authority,* **by means of which a theorem is proved.**

Every demonstration which does not meet the requirements of this definition, is defective and must be changed.

8. A Figure. In order that a theorem may be demonstrated, a representation of the things or relation of things named in its hypothesis is necessary. This representation is a line drawing called a figure.

Therefore **a figure is a representation by means of lines, of the thing or relation of things named in a theorem.**

9. Hypothesis and Conclusion. A theorem consists of two parts:

1. The given conditions.
2. The result of the specified conditions.

A statement of the first, is called the hypothesis; a statement of the second is called the conclusion. Therefore the following definitions:

The Hypothesis is a statement of the thing or relation of things represented in the figure.

The Conclusion is a statement of what is to be proved regarding the thing or relation of things named in the hypothesis.

10. Remark. Following are instructions for keeping the work-book, the list of abbreviations and symbols, definitions, axioms, and directions for writing an hypothesis and a demonstration.

After the instructions regarding the work-book have been read, the student is prepared to do the work on page 24.

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TECHNICAL GEOMETRY

THE WORK-BOOK

1. Description. The constructive form of this text requires that each student shall prepare the work daily in accordance with the suggestions, questions, and directions in numerous developing exercises. These exercises force a student not only to do his own thinking but to express his thought in written form.

For this purpose the mathematics work-book is used, which both in size and form is the result of seventeen years of experiment with over 2000 students. It consists of twelve daily record sheets, described in a subsequent paragraph, and 250 removable sheets of 16-pound, unruled, linen paper measuring $5\frac{1}{2} \times 8\frac{1}{2}$ inches, with fasteners for attaching the sheets to the back cover and the student's written work to the front cover.

2. Instruments. In order to prepare the work in a satisfactory manner and to secure the greatest educational benefit, each student will require the following equipment:

blotter,
12-inch triangular scale with U. S. and metric graduations,
medium lead pencil,
ink and pencil erasers,
fountain pen,
ruling pen,
compasses,
red, black, and India inks.

3. Value of Careful Work. It is obvious that skill is never acquired by careless, indifferent effort. Therefore, as in shop-work, the desideratum in this text is perfection in the finished work.

The instructions in the subsequent paragraphs and throughout the text, have accordingly been written to inspire each student with an enlarging ideal and to develop an increasing ability and enthusiasm for its realization.

4. Special Instructions for Work-Book Entries. Enter the **date** when work is done, in the upper right corner of the page.

Number the **pages** of the work-book successively as completed, in the lower right corner.

Write the work in **straight lines** uniformly spaced. Do not spread it either in the line or between lines.

If unable to write in a straight line, rule a sheet of the work-book in India ink, with lines from three-sixteenths to one-quarter of an inch apart and place this sheet under the page which is in preparation.

Center all **headings** with reference to margins, and keep them properly proportioned and spaced.

The work will look better and be more easily read if additional **space** is left between the conclusion and the demonstration, and between the demonstration and the theorem following the demonstration.

Make parentheses, equality signs, and other **symbols** carefully.

Letter **title-pages** in India ink without punctuation, for the subject and for each book, corresponding to the titles in the text. Insert these in the work-book preceded by a blank sheet.

Drawings in the development exercises should be drawn lightly and carefully with pencil and straight-edge, and after measurement, should be inked in with India ink and a ruling pen. Other drawings may be done directly in India ink.

As soon as possible learn to draw a light, smooth, draftsman's line.

5. Instructions for the Record Sheet. One dozen daily record sheets are furnished with the work-book. These have columns for the instructor's stamp and for the daily entry by the student, of date, paragraph and theorem numbers, and number of hours spent in outside preparation of studies.

Submit work for inspection as follows: On the first day prepare a record sheet by filling in the blanks as indicated. At the top of the time columns letter the names of the studies in which outside preparation is required, as Math, Phys, C L (chemistry laboratory), etc.

Observe that while the record sheet is for mathematics only, it is a time sheet for all studies.

Whenever work is submitted, attach it to the record sheet on which fill in the entries denoted by the column headings, making no entries in the remarks column.

When an additional record sheet is needed, place it on top of those already filled.

6. Excuse for Non-Performance and Absence. If unable to do assigned work, give to the instructor at the beginning of the period, a written excuse with

date,
exact reason for failure, and
signature.

Make the usual record sheet entries and write the word " Excuse " in the remarks column.

In case of absence enter the date of each day's absence and write " Absent " in the remarks column.

7. Collection and Distribution of Work-Books. (*a*) **Collection.** On the stroke of the bell at the beginning of the period, each student will pass his work-book along the row in reverse order from which the chairs or desks are numbered, being sure to place it on top of the books passed to him.

The student receiving the books in the last row, will collect each row's books and will place them in the file.

Work-books may be taken from the classroom only when permission is noted on the record sheet by the instructor.

(*b*) **Distribution.** At the beginning of the mathematics period, the collector will place the books at the end of the rows so that each student may receive his book from the pile as it is passed.

Books of absentees will be reported directly to the instructor's desk by the collector. In the collector's absence, the next student in the row will attend to the books.

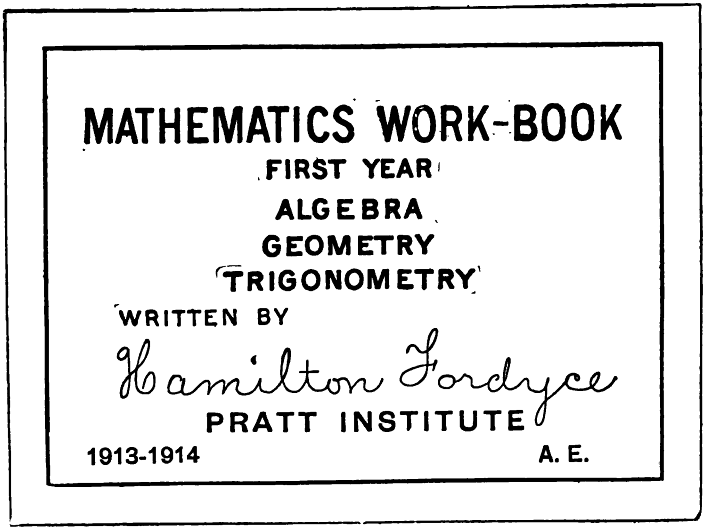
8. Inspection. The remarks column on the record sheet is for the instructor's stamp. When the dater is used instead of the " accepted " stamp, it signifies that the work is incomplete, or unsatisfactory, or incorrect. Changes in such work, unless obvious or indicated in the book, must be arranged with the instructor before the close of the period.

9. Corrected Work. Incorrect or rejected work is due in correct form at the beginning of the next mathematics' period.

Make corrections in red ink, on the same page with the incorrect work. If numerous mistakes have been made prepare a new page and insert it following the incorrect one, on which write in red ink " Corrected on next page," with date of correction.

10. Index and Label. At the end of the school year arrange all the work-books of the year in order, make an **alphabetic index** for your combined book, cut the work-book cover in two through the back, if necessary, and bind all together with one pair of fasteners.

On the front cover, attach a **label** about 4×5, with a line border and lettered title, enumerating the subjects covered, as suggested in the following model:


MATHEMATICS WORK-BOOK
FIRST YEAR
ALGEBRA
GEOMETRY
TRIGONOMETRY
WRITTEN BY
Hamilton Fordyce
PRATT INSTITUTE
1913-1914 A. E.

11. Development. In writing the development, use no rippled lines or authorities.

Paragraph each statement.

For accuracy of measurement, do all drawing in development exercises in hair-lines and make the parts sufficiently large to permit the ready use of scale and protractor.

Lines can be measured with greatest certainty by taking their lengths in the dividers and by applying the dividers to the scale.

Drawings must be measured before being inked in.

In every development exercise, *state exactly what the drawing represents and enter all measurements on the drawing.*

Do not letter the drawing unless absolutely necessary for clearness of statement in the development.

12. Hypothesis. Begin an hypothesis with the word **given.** Then specify the first thing named in the theorem, followed by the remaining conditions in the same order as in the theorem.

Following each condition, write the name of its linear representative in the figure. Omit no conditions expressed in the theorem and specify no others.

Take great pains that hypotheses are correctly expressed. A common fault is to name conditions in the hypothesis without regard to their order in the theorem. This leads to loose thinking and sometimes to omissions. That this may not occur, write hypotheses as follows:

(1) The word Given.
(2) The *theorem* name of the first specified thing in the theorem.
(3) The *figure* name of the first thing specified in the theorem.
(4) The theorem name of any other conditions of the theorem, in their order, each followed by its figure name.

13. Demonstration. Write the demonstration immediately following the conclusion, *separated* from it *by a double space.* It should begin about half or three-quarters of an inch from the left margin.

Number every statement of a demonstration at the left consecutively, in Arabic notation without punctuation, inclosing the number in a small parenthesis.

Number each successive statement of the demonstration, at the same distance from the margin.

After each statement, make a short rippled line, followed by the mathematical authority for the statement.

Designate authorities by abbreviations and symbols, and not by numbers.

Close each demonstration by the word therefore, followed by the *exact words of the theorem.*

Leave an extra space both before and after the demonstration.

No statement of the demonstration should occupy more than about two-thirds the width of the page. If greater space is needed, use an extra line.

Use abbreviations and symbols whenever possible.

Do not inclose authorities in parenthesis.

Do not make rippled lines shorter than three-fourths of an inch long. Longer lines will sometimes be desirable.

14. Authorities. Each statement of a demonstration must be followed by the mathematical authority on which the statement is based. Of these there are *four* only:

Hypothesis,
Axioms,
Definitions,
Theorems previously demonstrated.

No statement which is not based on one or more of these authorities, can be regarded as a part of the demonstration.

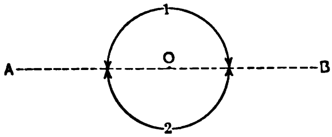
Indicate authorities invariably by symbols and abbreviations, written on the same line as the statement which they substantiate. To avoid confusion separate them from the statements of demonstration, by space or a rippled line as shown in the model demonstration.

Statements with space or rippled lines following, should not occupy more than two-thirds of the width of the page, the *remaining third being reserved for authorities.*

15. Model Demonstration. The following is recommended as the most desirable way of writing a demonstration, the model shown being a demonstration of the theorem that the sum of the angles about a point equals 360°.

Observe the numbering, the rippled lines, the extra space preceding and following the demonstration, the symbols, and the indented hypothesis, conclusion, and theorem.

The sum of the angles about a point equals 360°.



Given Pt O
Prove Sum ∠s Abt $O = 360°$

(1) Thru O draw line AB } ∿∿∿∿∿∿ Cons Ax
Mkg ∠s 1 and 2 }

(2) Then $1 = 180°$ } ∿∿∿∿∿∿ { Def St ∠
And $2 = 180°$ } { & St ∠ $= 180°$

(3) ∴ $1 + 2 = 360°$ ∿∿∿∿∿∿ Add Ax

(4) But $1 + 2 =$ Sum ∠s Abt O ∿∿∿ Sum Pts Ax

(5) ∴ Sum ∠s Abt $O = 360°$ ∿∿∿∿ =ity Ax

Therefore the sum of the angles about a point equals 360°.

General Axioms

1. **=ty Ax.** Things equal to the same thing or equal things are equal to each other.

2. **Add Ax.** If equals are added to the same thing or to equals, the sums are equal.

3. **Sub Ax.** If equals are subtracted from the same thing or from equals, the remainders are equal.

4. **Mul Ax.** If equals are multiplied by the same thing or by equals, the products are equal.

5. **Div Ax.** If equals are divided by the same thing except zero, or by equals, the quotients are equal.

6. **In=ity Ax Add.** If unequals are added to the same thing or to equals, the sums are unequal and the greater sum is the one to which the greater addition is made.

7. **In=ity Ax Sub.** If unequals are subtracted from the same thing or from equals, the remainders are unequal and the greater remainder is the one from which the lesser subtraction is made.

8. **In=ity Ax Mul.** If unequals are multiplied by the same thing or by equals, the products are unequal and the greater product is the one obtained from the greater multiplicand.

9. **In=ity Ax Div.** If unequals are divided by the same thing or by equals, the quotients are unequal and the greater quotient is the one obtained from the greater dividend.

10. **Sum Pts Ax.** The whole equals the sum of its parts and is therefore greater than any of its parts when all are positive.

11. **Power and Root Ax.** Like powers and like roots of equals are equal.

12. **Frac Ax.** If both numerator and denominator of a fraction are multiplied or divided by the same thing, the value of the fraction is unchanged.

13. **Subs Ax** * (**Substitution Axiom**). A quantity may be substituted for its equal, in any equation or inequality, without affecting the relation of the members.

Geometric Axioms

14. **Bet 2 Pts 1 St.** Only one straight line can be drawn between two points. Two straight lines can therefore intersect in one point only.

15. **St Line Shortest.** A straight line is the shortest line between two points. Two straight lines whose extremities coincide are therefore equal.

16. **Thru Pt 1** ‖. Only one straight line can be drawn through a point, parallel to a given straight line.

17. **Super Ax** (**Superposition Axiom**). Any geometric figure or any part of it may be rotated, revolved, or otherwise moved in space, provided the essential relation of parts used in demonstration is unaffected.

18. **Cons Ax** * (**Construction Axiom**). A perpendicular, or a parallel, may be drawn or produced. Any other construction necessary to demonstration and not contrary to hypothesis may be made.

19. **Z Ax** * (**Coincidence Axiom**). (1) Any line is in the same relation to another line, directionally, as the line with which it coincides.

(2) A line passes through the same point as the line with which it coincides.

20. **Ex Ax** * (**Exclusion Axiom**). A relationship based

---

* See note, page 11.

on an assumption which results in a condition contrary to hypothesis or definition, does not exist.

21. **Pos Ax * (Position Axiom).** A point *is* or is *not*, in a line.

22. **Iden Ax * (Identity Axiom).** Any quantity is equal to itself.

---

* The author has found it possible to demonstrate certain theorems common to all geometries, only by the use of axioms hitherto unformulated. These are:

The substitution axiom,
The construction axiom,
The coincidence axiom,
The exclusion axiom,
The position axiom,
The identity axiom,
The magnitude axiom,
The perpendicularity axiom,
The parallelism † axiom.

The first six are formulated above, as axioms 13, and 18 to 22 inclusive; the remainder are developed in the text as required in demonstration.

Every teacher and student of geometry has tacitly assumed these axioms but the statements of demonstration for which they are the only authority, have hitherto been unsubstantiated.

† To TEACHERS. As will be seen this is not the parallel axiom of Euclid.

ABBREVIATIONS

Alt Int	alternate-interior	Interc	intercept
Asm	assume	Isos	isosceles
Asmg	assuming	Lim	limit
Ax	axiom	Mkg	making
Bis	bisect	Obl	oblique
Coms	commensurable	Opp	opposite
Concl	conclusion	Pl	plane
Cons	construction	Poly	polygon
Const	constant	Pro	proportion
Cor	corollary	Prob	problem
Cyl	cylinder	Prop	proposition
Def	definition	Psb	possible
Defs	definitions	Quad	quadrilateral
Descr	describe	Rad	radius
Diag	diagonal	Rat	ratio
Diam	diameter	Sec	secant
Ext Int	exterior-interior	St	straight
Fig	figure	Sub	subtract
Figs	figures	Subs	substitute
Fr	from	Super	superpose
Homol	homologous	Supp Adj	supplementary-adjacent
Hyp	hypothesis	Tan	tangent
Iden	identical, identity	Theo	theorem
Impsb	impossible	Trap	trapezoid
Incl	included	Val	value
Incoms	incommensurable	Var	variable
Inscr	inscribed		

This list might be extended but these are sufficient to show the principle of mathematical abbreviation. In general it is to use the first syllable.

Observe that each abbreviation begins with a capital and is *not* followed by a period.

SYMBOLS

∠	angle	▱	parallelogram
∠s	angles	▱s	parallelograms
⌒	arc	∥eπ	parallelepiped
⌒s	arcs	⊥	perpendicular
⊙	circle	⊥s	perpendiculars
⊙s	circles	▭	rectangle
≎	is equivalent to	▭s	rectangles
>	is greater than	Rt ∠ or ∟R	right angle
<	is less than	Rt ∠s or ∟s	right angles
∥	is parallel to, or parallel	∼	is similar to or are similar
∥s	parallels	△	triangle
∦*	is or are non-parallel	△s	triangles
≠*	is or are unequal	∴	therefore

≐ approaches as a limit

The following symbols, formed by conventionalizing respectively, *S* in the word coincides and *D* in the word measured, are suggested for use:

Ƶ coincides with.

⊐ is measured by.

---

[*] An oblique line through a symbol negatives the idea which the symbol represents.

DEFINITIONS *

1. **An Axiom** is a self-evident, undemonstrable truth.

2. **A Theorem** is a truth admitting of, or requiring, demonstration.

3. **A Figure** is a linear representation of the thing or relation of things, named in a theorem.

4. **A Hypothesis** is a statement of what thing or relation of things, is given in a theorem.

5. **A Conclusion** is a statement of what is to be proved regarding the thing or things named in a hypothesis.

6. **A Demonstration** is a logically arranged series of statements, each based on mathematical authority, by means of which a theorem is proved.

7. **A Plane** is a surface containing every point on a straight line through any two of its points.

8. **A Plane Figure** is a figure all of whose points are in one plane.

9. **An Angle** is a plane figure formed by two lines which meet at a point. The point is the vertex of the angle; the lines are the sides of the angle.

10. **A Straight † Line** is a line which is completely determined by any two of its points.

11. **A Straight Angle** is an angle whose sides are in the same straight line with its vertex, but in opposite directions.

---

* The definitions in this list are in the order required in the demonstration of the theorems in plane geometry. An alphabetic list for both plane and solid is at the end of this text.

† Unless otherwise specified, line means straight line.

12. **A Right Angle** is an angle formed by a vertical line meeting a horizontal line, or by a line making equal adjacent angles with another line.

13. **An Acute Angle** is an angle less than a right angle.

14. **An Obtuse Angle** is an angle greater than a right angle and less than a straight angle.

15. **A Degree** is an angle which is one-ninetieth of a right angle. A right angle therefore, equals ninety degrees. (The degree is the unit of angular measure.)

16. **A Perpendicular** is a line which forms a right angle with another line, i.e., an angle of 90 degrees.

17. **Adjacent Angles** are angles having a common side, a common vertex, and two sides exterior.

18. **The Exterior Sides** of two adjacent angles are the outer sides, i.e., the sides which lie on opposite sides of the common side.

19. **Supplementary Angles** are angles whose sum equals 180°. The supplement of an angle is therefore the difference between 180° and the angle.

20. **Intersecting Lines** are lines which cross each other.

21. **Vertical Angles** are the pairs of opposite angles formed by two intersecting lines.

22. **Equal Angles** are angles which coincide or will coincide when in the same plane with one side and the vertex common, and one angle on the other.

23. **Equal Lines** are lines of the same length, i.e., lines which are equal extensionally. They therefore coincide or can be made to coincide directionally and extensionally.

24. **An Oblique Line** is a line which makes neither a right angle nor a straight angle with another line.

25. **Parallel Lines** are lines in the same plane, which will not meet within a finite distance, even when produced.

26. **Non-Parallel Lines** are lines which meet if produced in the direction of their convergence.

27. **Alternate-Interior Angles** are angles formed by two lines and a transversal, the angles being non-adjacent, within the two lines, and on opposite sides of the transversal.

28. **Exterior-Interior Angles** are angles formed by two lines and a transversal, the angles being non-adjacent, on the same side of the transversal, one without the two lines, the other within.

29. **A Triangle** is a plane figure bounded by three lines.

30. **A Polygon** is a plane figure bounded by three or more lines.

31. **An Exterior Angle of a Triangle** is an angle formed without the triangle, by any side and a produced side.

32. **Superposition** is the movement of any part of a figure, or the placing of one figure in coincidence with any part of another in its plane.

33. **Equal Triangles** are triangles whose homologous sides are equal and whose homologous angles are equal; they therefore coincide or can be made to coincide, i.e., they have the same size and shape.

34. **Coincident Figures** are figures whose points are identical.

35. **Equal Figures** are figures which coincide or can be superposed so as to coincide.

36. **Homologous Sides** are sides opposite equal angles in equal or similar triangles.

37. **Homologous Angles** are angles opposite equal sides in equal triangles, or opposite corresponding sides in similar triangles.

38. **A Corollary** is a theorem easily deduced from another theorem.

39. **An Isosceles Triangle** is a triangle having two equal sides. The equal sides are called legs.

40. **The Vertex of a Triangle** is the vertex of the angle opposite the base.

41. **A Right Triangle** is a triangle having one right angle.

42. **A Quadrilateral** is a polygon having four sides.

43. **A Diagonal of a Quadrilateral** is a line joining any two opposite vertices.

44. **A Parallelogram** is a quadrilateral whose opposite sides are parallel.

45. **A Trapezoid** is a quadrilateral two of whose sides are parallel and whose other two sides are not parallel.

46. **A Median of a Triangle** is a line joining any vertex to the middle of the opposite side.

47. **A Circle** is a plane figure from whose center all points of its boundary line are equidistant.

48. **The Circumference of a Circle** is its boundary line.

49. **A Chord** is a line whose extremities are in the circumference.

50. **A Diameter** is a chord through the center of a circle.

51. **A Radius** is a line from the center to the circumference of a circle.

52. **A Central Angle** is an angle whose vertex is at the center of a circle and whose sides are radii.

53. **An Arc** is any part of the circumference.

54. **A Tangent to a Circle** is a line, all of whose points except one, the point of tangency, are without the circumference even when the line is produced.

55. **Equal Circles** are circles whose radii are equal.

56. **A Line of Centers** is a line joining the centers of circles.

57. **A Secant** is a line having two points in the circumference and cutting the circumference in one or both of the points.

58. **Tangent Circles** are circles whose circumferences have only one point in common.

59. **Commensurable Quantities** are quantities which are evenly divisible by the same unit of measure.

60. **The Ratio of Two Commensurable Quantities of the Same Kind** is their relative magnitude expressed as a fraction, the numerator being the first quantity, the denominator the second quantity.

61. **An Inscribed Angle** is an angle whose vertex is in the circumference and whose sides are chords of a circle.

62. **An Inscribed Polygon** is a polygon whose vertices are in the circumference of a circle and whose sides are therefore chords.

63. **An Angle Measured by its Intercepted Arc** is an angle whose number of angle degrees equals the number of intercepted arc degrees.

64. **A Proportion** is an equality of ratios.

65. **The Terms of a Proportion** are the quantities forming the proportion.

66. **The Means** are the second and third terms, i.e., the intermediate terms.

67. **The Extremes** are the first and fourth terms.

68. **The Antecedents** are the first terms of each ratio.

69. **The Consequents** are the last terms of each ratio.

Mathematics Work-book in which each student prepares all assigned work.

70. **A Mean Proportional** is a quantity which forms the means of a proportion of three quantities, i.e., a quantity which is in the same ratio to one quantity that it is to another inversely.

71. **Alternation** is the interchange of the means or the extremes of a proportion.

72. **Inversion** is the interchange of the means and the extremes of a proportion, i.e., an inversion of the ratios.

73. **Composition** is the compounding of the ratios of a proportion; i.e., the sum of the first and the second term, is to the first or the second, as the sum of the third and the fourth term, is to the third or the fourth.

74. **Division** is the dividing of the ratios of a proportion; i.e., the first minus the second term, is to the first or the second, as the third minus the fourth term, is to the third or the fourth.

75. **Composition and Division** is the compounding and the dividing of the ratios of a proportion; i.e., the sum of the first and the second term, is to the difference of the first and the second term, as the sum of the third and the fourth term, is to the difference of the third and the fourth term.

76. **Equimultiples** of two quantities are products obtained by multiplying each of the quantities by the same quantity.

77. **A Continued Proportion** is a proportion of three or more ratios.

78. **Similar Triangles** are triangles whose corresponding angles are equal and whose homologous sides are proportional.

79. **Similar Polygons** are polygons whose corresponding angles are equal and whose corresponding sides are proportional.

80. **Proportional Quantities** are quantities which form a proportion.

81. **An Altitude of a Triangle** is a perpendicular from the vertex of any angle to the opposite side.

82. **The Perimeter of a Figure** is the length of its boundary line; i.e., the distance around it.

83. **The Projection of a Line upon Another** is the segment of the second line included between the feet of the perpendiculars to that line from the extremities of the given line.

The projection upon the other therefore, of one of two lines which meet but do not intersect, as line *a* upon line *b*, is the segment of *b* included between the common point of *b* and *a*, and the foot of the perpendicular to *b* from the other extremity of *a*.

84. **A Rectangle** is a right-angled parallelogram.

85. **The Base of a Polygon** is the side upon which it stands or may be assumed to stand.

86. **The Altitude of a Parallelogram** is the perpendicular distance from the base to the opposite side.

87. **A Unit of Area** is a square whose side is one linear unit.

88. **The Area of a Polygon** is its ratio to the unit of area; i.e., the ratio of its surface to the surface of the unit of area.

89. **A Square** is an equilateral rectangle.

90. **The Product of Two Lines** is the product of their numerical measures.

91. **The Algebraic Square of a Line** is the square of its numerical measure.

92. **The Geometric Square of a Line** is the square described on the line as a side.

93. **A Line Divided in Extreme and Mean Ratio** is a line divided into two unequal segments, the greater of which is a mean proportional to the whole line and the lesser segment.

94. **Equivalent Figures** are figures of the same size; i.e., figures of equal area.

95. **A Regular Polygon** is a polygon which is equilateral and equiangular.

96. **The Center of a Regular Polygon** is a point equally distant from the sides of a polygon.

97. **The Radius of a Regular Polygon** is the distance from the center to any vertex.

98. **The Apothem of a Regular Polygon** is the distance from the center to any side.

99. **Concentric Circles** are circles having the same center.

100. **A Circumscribed Polygon** is a polygon whose sides are tangent to a circle. The circle is inscribed.

101. **An Angle at the Center of a Regular Polygon** is an angle formed by radii to the extremities of any side.

102. **A Sector** is a portion of a circle included by two radii and the intercepted arc.

103. **A Segment** is a portion of a circle included by a chord and its subtended arc.

104. **Similar Arcs** are arcs of the same circle or of different circles, intercepted by equal angles at the center.

105. **Similar Sectors** are sectors of the same circle or of different circles, whose arcs are intercepted by equal angles at the center.

106. **Similar Segments** are segments of the same circle or different circles, whose arcs intercept equal angles at the center.

107. **A Pentagon** is a polygon of five sides.

108. **A Hexagon** is a polygon of six sides.

109. **A Heptagon** is a polygon of seven sides.

110. **An Octagon** is a polygon of eight sides.

111. **An Enneagon or Nonagon** is a polygon of nine sides.

112. **A Decagon** is a polygon of ten sides.

113. **An Endecagon** is a polygon of eleven sides.

114. **A Dodecagon** is a polygon of twelve sides.

115. **A Pentedecagon** is a polygon of fifteen sides.

116. **A Unit Circle** is a circle whose radius is unity.

117. **Incommensurable Quantities** are quantities which have no common measure and are therefore not evenly divisible by the same quantity.

118. **A Constant** is a quantity whose value does not change.

119. **A Variable** is a quantity whose value changes continually.

120. **A Limit** (geometric) is (1) a constant; (2) the constant which a variable continually approaches; and (3) the constant from whose value the variable may ultimately be made to differ by less than any assignable quantity.

BOOK I

STRAIGHT LINES AND ANGLES

1. St $\angle = 180°$

2. Ext Sds St, $\angle$s Supp
3. Adj $\angle$s Supp, Ext Sds St

4. Vert $\angle$s $=$

5. At Pt 1 $\perp$
6. Fr Pt 1 $\perp$
7. Pt in $\perp$ Bis, $=$ly
8. Pt without $\perp$ Bis$\neq$ly
9. $\perp$ Shortest

10. Sum 2 Lines $>$

11. Dists$=$, Lines$=$
12. Dists$\neq$, more remote Gr
13. Lines$=$Dists$=$
14. Lines$\neq$greater cuts Gr Dist

15. Lines $\perp$ same
16. Line $\perp$ one of 2 $\parallel$s

17. Lines $\parallel$ same

18. Alt Int $\angle$s
19. Alt Int$=$ $\parallel$

20. Ext Int $\angle$s
21. Ext Int$=$ $\parallel$

22. Sum 2 Int $\angle$s
23. Sum 2 Int$=180°$ $\parallel$

24. Alt Ext $\angle$s
25. Alt Ext$=$ $\parallel$

26. $\angle$s Sds Resp $\parallel$
27. $\angle$s Sds Resp $\perp$

The spacing in this outline is used to indicate the grouping of related theorems.

Whenever in demonstration these theorems are cited as authorities, they should be indicated as here outlined.

I THEOREM 1

DEVELOPMENT. (See page 5.) About one and one-half inches from the top of the page of the work-book, midway between margins, write book number and theorem number without punctuation as shown above.

Observe that the book number is in Roman notation, the theorem number in Arabic.

On the left margin about one-half inch below the heading, letter the word development.

On the next line midway between margins locate a point *C*, placing the letter either above or below the point.

From *C* draw two lines *CA* and *CD* in opposite directions, not less than two inches in length.

State what the drawing represents.

CA and *CD* form what kind of an angle?* (See Def 11.)

Why? (Express answer in the words of the definition.)

Measure $\angle ACD$ with the protractor.

$\angle ACD =$ how many degrees?

Therefore what kind of angle equals how many degrees?

In beginning the study of geometry, we must not depend on common knowledge but must *know* only from two sources:

1. Construction, definition, measurement.
2. Demonstration.

---

* In answering all questions, include the question in the answer.

Demonstration. Letter this heading on the left margin and below write as a theorem your answer to the last question, using no symbols or abbreviations.

About an inch below the theorem and midway between margins, draw the figure, for convenience making the sides of the angle about two inches long. (See 8, page x.)

Letter *C* at the vertex and *A* and *D* at the extremities of the sides.

Immediately under the figure write the following hypothesis and conclusion:

Given St $\angle ACD$
Prove $\angle ACD = 180°$.

Write the demonstration as follows, immediately following the conclusion: (See pages 6–8.)

State what kind of an angle *ACD* is. (See Hyp.)
$\therefore$ *AD* is what kind of a line? (See Def 11.)
Draw $CX \perp AD$ Mkg $\angle$s 1 and 2.[*] (Write this statement in your demonstration.)
Then $1 =$ how many degrees?
And $2 =$ how many degrees? } (See Def 16.)
$\therefore$ $1+2 =$ how many degrees?
But $1+2 =$ what other angle? (See Ax 10.)
$\therefore$ $\angle ACD =$ how many degrees?

Therefore a straight angle, etc. (See page 7.)

---

[*] Construction lines of dashes are used in geometry to make a demonstration possible but never to represent the specifications of a theorem. This distinction must be strictly observed.

Explanation. We now know that a straight angle equals 180°, by two different methods:

1. Construction and measurement.
2. Demonstration.

These methods of obtaining definite knowledge are entirely distinct and must be so regarded in subsequent work.

The first is called graphical, constructive, or inventive geometry; the second demonstrative or Euclidian geometry.

By plane geometry or merely geometry, is meant demonstrative geometry.

In this book both methods are used, the first as a means to the discovery of the theorem which is to be demonstrated.

Although facts established by the first method are just as positive as by the second, none obtained in this way must be used as authority in demonstration.

In other words, in this book construction and measurement are employed to discover demonstrable truths regarding the space in which we live; in plane geometry, space of two dimensions, in solid geometry, space of three dimensions.

Truths thus discovered are expressed formally as theorems, and through demonstration are known mathematically.

Of the three ways of knowing, (1) by experience, (2) by the senses, and (3) by demonstration, only the last is recognized in demonstrative geometry.

What is demonstration? It is to prove a theorem, not by measurement or by the eye or by common knowledge, but by **mathematical authorities.** These are four only:

Hypothesis,
Definitions,
Axioms,
Theorems previously demonstrated.

Unless based on one or more of these authorities, no statement of a demonstration is valid.

To illustrate, consider the demonstration of theorem 1.

Before demonstration we knew *mathematically* only the facts of the theorem, namely that a straight angle was given, and that we must prove it equal to 180°.

By hypothesis, then, we had a straight angle.

By definition, we knew its sides formed a straight line.

By the authority of the construction axiom, we drew a perpendicular to this line at the vertex of the angle.

By definition of a perpendicular, we knew each angle formed by this perpendicular equaled 90°.

But 180° was wanted.

By the authority of the addition axiom, we therefore added the two equations, obtaining 180°.

But we wished to show the straight angle equal to 180°.

Therefore by the sum of the parts axiom, we knew the sum of the two angles equaled the straight angle.

We now had the straight angle and 180°, both equal to the sum of the two angles.

Therefore by the equality axiom, we knew the straight angle was equal to 180°.

The mathematical authorities for the demonstration were hypothesis, definition of straight angle, construction axiom, definition of perpendicular, the addition, the sum of the parts, and the equality axioms.

No reference was made to the graphical work.

It should be noted that *the conclusion is not mathematical authority, and therefore must never be employed in demonstration.* Its only value consists in its statement of what is to be proved.

In demonstrative geometry, the first problem is to determine what is known *mathematically* from hypothesis; the second, is to use this knowledge in demonstration.

I THEOREM 2

DEVELOPMENT. Enter the headings as in the preceding theorem.

Draw two adjacent angles whose exterior sides are *not* in a straight line.

Draw two whose exterior sides are in a straight line.

State what the drawings represent.

Are the first two angles supplementary? (See definition.)

Are the last two? Why?

Therefore adjacent angles whose exterior sides are in a straight line, are in what relation?

DEMONSTRATION. Write as a theorem your answer to the last question.

Draw the figure.

Name the angles 1 and 2.

Write the hypothesis.

Write the conclusion.

Write the demonstration, answering the following questions:

AB is what kind of line? (See hyp.)

$\therefore$ $\angle ACB$ is what kind of an angle? (See Def 11.)

$\therefore$ $\angle ACB$ equals how many degrees? (See Theo 1.)

(Write your answer as an equation.)

But $\angle ACB$ equals the sum of what other two angles? (See axiom 10.)

(Write your answer as an equation.)

$\therefore$ $1+2=$how many degrees?

$\therefore$ 1 and 2 are what? (See Def 19.)

I THEOREM 3

DEVELOPMENT. Draw two supplementary adjacent angles.

State what the drawing represents.

Their sides form what kind of a line?

Therefore supplementary adjacent angles have their exterior sides in what kind of a line?

DEMONSTRATION. Write as a theorem your answer to the preceding question.

Draw the figure, naming the angles as in theorem 2.

State the hypothesis and the conclusion.

1 and 2 are in what relation? (See hyp.)

Therefore $1+2=$ how many degrees?

But $1+2=$ what other angle?

Therefore $\angle ACB=$ how many degrees?

Therefore $\angle ACB$ is what kind of an angle? (See Theo 1.)

Therefore AB is what kind of a line?

"The better part of man is the reason, which must therefore be the chief object in education."—ARISTOTLE.

"The understanding is not a vessel that needs filling; it is fuel that needs kindling. It is kindled by truth."—PLUTARCH.

I THEOREM 4

DEVELOPMENT. Draw vertical angles.

State what the drawing represents.

By the protractor, determine and state the relation of each angle to its opposite.

Dimension the measurements on the drawing.

DEMONSTRATION. Write a theorem specifying the relation of angles of this kind, beginning the theorem with the name of the angles.

Draw the figure, naming one pair of angles 1 and 2, and the other pair 3 and 4.

Write the hypothesis and the conclusion.

1 and 3 have their Ext Sds in what kind of a line? (See Hyp and Def Vert $\angle$s.)

Therefore 1 and 3 are in what relation? (See Theo 2.)

Therefore $1+3=$ how many degrees?

Write the proof that $3+2$ equals the same number of degrees, and apply the equality axiom.

Solve the resulting equation for either angle desired.

In the same way prove $3=4$, entering every statement.

"Permit the pupil himself to taste and relish things, and of himself to choose and discern them, sometimes opening the way for him, sometimes leaving him to break the ice for himself."
—MONTAIGNE.

I THEOREM 5

DEVELOPMENT. Draw a straight line.

At any point in the line, draw a perpendicular to the line.

State what the drawing represents.

At the same point draw another perpendicular if possible, to the same line.

Therefore at a point in a line, only how many perpendiculars can be drawn?

DEMONSTRATION. Write as a theorem your answer to the preceding question.

Draw the figure representing *O* as a point in the line *AB*, and *OC* perpendicular to *AB*.

State the hypothesis.

Write the conclusion as follows:

Prove *OC* only Psb ⊥ *AB* at *O*.

Assuming it possible, draw *OX* any other perpendicular to *AB* at *O*.

(Write this as the first statement of the demonstration, using abbreviations and symbols.)

OC is in what relation to *AB*?

Therefore ∠ *AOC* equals how many degrees?

But *OX* is in what assumed relation to *AB*?

Therefore ∠ *AOX* equals how many degrees?

Therefore what is the relation of ∠ *AOC* to ∠ *AOX*?

Therefore *OX* coincides with what? (Def 21.)

Therefore what line is the only possible perpendicular to *AB* at *O*? (Ax.)

I THEOREM 6

DEVELOPMENT. Represent a point *without* (not in) a straight line.

Draw a perpendicular from the point to the line.

State what the drawing represents.

If possible, draw another perpendicular to the line from the same point.

Therefore from a point, only how many perpendiculars can be drawn to a line.

DEMONSTRATION. Write your last statement as a theorem.

Draw the figure, making *O* the point, *AB* the line, and *OX* the perpendicular to *AB*.

Write the hypothesis.

Write the conclusion as follows:

Prove OX only Psb $\perp$ AB fr O.

Assuming it possible, draw *OR*, any other perpendicular to *AB* from *O*.

On *AB* revolve *XOR* into the plane of the figure on the opposite side of *AB*, denoting by *F* the new position of *O*.

(In the demonstration state this as follows:

Superpose *XOR* into position *XFR*.)

Then $\angle RXF$ = how many degrees?

And $\angle XRF$ = how many degrees?

Superpose *XOR* into its original position.

Prove *OXF* a straight line.

Prove *ORF* a straight line.

∴ Is *OX* the only possible perpendicular from *O* to *AB*?

I THEOREM 7

DEVELOPMENT. Represent a point *in* a perpendicular at the middle of a line.

State what the drawing represents.

Determine by measurement and state whether the point is unequally or equally distant from the extremities of the line.

Enter your measurements.

Are all other points in the perpendicular equally distant from A and B?

DEMONSTRATION. Write the theorem beginning, Any point, etc.

Draw the figure, representing V as any point in OS perpendicular to AB at the middle point S, and VB and VA the distances.*

State† the relation of $\angle VSA$ to $\angle VSB$.

On OS revolve one part of the figure until SB falls on SA. (Super Ax and Def 22.)

Write this statement in your demonstration.

Therefore on what point will B fall? (Hyp and Def 23.)

Therefore on what point will VB fall? (Ax 14.)

Therefore what is the relation of VB to VA? (Def 23.)

---

* By distance, as used in geometry, is meant the shortest distance.

† In all subsequent work, be sure to write hypothesis and conclusion before beginning the demonstration. In this instance write the conclusion as follows: Prove $VA = VB$.

I THEOREM 8

DEVELOPMENT. Represent a point *without* a perpendicular at the middle of a line.

Is the point unequally or equally distant from the extremities of the line? (Enter measurements.)

DEMONSTRATION. Write the theorem.

Letter the figure as in theorem 7, taking V on the same side of the perpendicular as the point B.

Connect B with X, the point of intersection of VA and OS. (In your demonstration write, Draw BX.)

What is the relation of $VX+XB$ to VB? (Ax 15. See page 13 for symbol.)

But XB equals what? (Theo 7.)

Therefore what is the relation of $VX+XA$ to VB? (Ax 13.)

SPECIAL SUGGESTION

An Inequality. In forming an inequality, care must be taken that the line which is to be proved greater, or at least some part of it, is written in the greater side of the inequality; also, that neither the line which is to be proved less nor any part of it, is written in the greater side.

In other words what is to be proved greater, must wholly or in part be written in the *greater* side; what is to be proved less, must wholly or in part be written in the *lesser* side.

Illustration. In the demonstration of theorem 8, it was necessary to prove VA greater than VB.

VA was in two segments, VX and XA.

The inequality was therefore written with one of these, *VX*, in the greater side.

But *VB* was to be proved less. It therefore could not be added to *VX* in the greater side, but had to be written in the lesser side.

I THEOREM 9

DEVELOPMENT. Represent a point without a line.

Draw three lines from the point to the line, one of them perpendicular.

State what the drawing represents.

Could all the lines from the point, be perpendicular? Why?

Measure each of the lines with the scale and enter the measurements.

Which is the shortest?

Would it be shortest of any which might be drawn from the point to the line?

DEMONSTRATION. Write a theorem regarding the shortest line from a point to a line.

Draw the figure, making *O* the point, *AB* the line, *OR* the perpendicular, and *OX* any other line from *O* to *AB*.

Produce *OR* to *Q* (on opposite side of *AB*) Mkg $OR = RQ$.

Draw *QX*.

Then $OR+RQ$ is in what extensional relation to $OX+XQ$?

But *RQ* equals what other line?

And *XQ* equals what other line? (Hyp, Cons and Theo 7.)

In the inequality substitute these values for *RQ* and *XQ*, and collect terms.

Finish the demonstration.

I THEOREM 10

DEVELOPMENT. Draw two lines from a point, to the extremities of a line.

Join the same extremities to a point included by the lines first drawn.

Measure the four lines from the extremities.

Which is greater, the sum of the two outer lines or the sum of the two inner lines?

DEMONSTRATION. Write a theorem based on this development.

Draw the figure with the lines OA and OB from O to the extremities of AB, and RA and RB the included lines.

Produce AR until it meets OB at X.

State the extensional relation of $AO+OX$ to $AR+RX$.

In like manner compare $BX+XR$ and BR.

Add the inequalities member to member.

By two operations change the resulting inequality to that given in the conclusion.

I THEOREM 11

DEVELOPMENT. Draw a perpendicular to a line at any point in the line.

From a point in the perpendicular draw two lines terminating in the line at equal distances from the foot of the perpendicular.

By measurement, determine and state the extensional relation of the two lines which were drawn from the point in the perpendicular.

DEMONSTRATION. Write as a theorem, a statement of the extensional relationship of two lines from a point in a perpendicular, which terminate in the line to which the perpendicular is drawn, at *equal* distances from the foot of the perpendicular.

Demonstrate by applying a preceding theorem.

I THEOREM 12

Write a theorem regarding the extensional relation of two lines from a point in a perpendicular to a line, which terminate in the line at *unequal* distances from the foot of the perpendicular, specifying which is greater.

(No development or lettered headings are necessary.)

Draw the figure, showing the lines OS and OV, EF the perpendicular, and FV greater than FS.

If your figure is not drawn so that both of the lines are on the same side of the perpendicular, lay off from the foot of the perpendicular the shorter distance on the greater.

Connect X, the point thus determined, with O.

State the relation of OX to OS.

Produce OF its own length through the line to which it is perpendicular, and connect its produced extremity R, with X and with V.

State the extensional relation of $OV+VR$ to $OX+XR$.

But VR is in what relation extensionally to OV?

And XR is in what extensional relation to OX?

Perform such other operations as may be necessary to complete the demonstration.

Axiom 23. Magnitude Axiom. Enter the heading in the work-book. Under it draw the following lines one below the other, not intersecting, each near the left margin. Following each write what the line is, as specified on page 38.

AB, a line of convenient length;
CD, equal to *AB*;
EF, greater than *AB*;
GH, less than *AB*. (Use the proper symbols.)

If possible draw a line in none of these extensional relations to *AB*.

Therefore what extensional relations, only, may one line sustain to another? (Enumerate in the same order as represented by the lines you have drawn.)

Therefore magnitudes are in what relations?

I THEOREM 13

Write a theorem based on the following question:

Two equal lines from a point in a perpendicular to a line, cut off distances from the foot of the perpendicular, in what extensional relation?

Draw the same figure as in theorem 12, with $OV = OS$, omitting the construction lines.

Demonstrate as follows:

Suppose *FS* less than *FV*. (Insert this statement in your demonstration. The authority is Mag Ax.)

Then what is the extensional relation of *OV* and *OS*? (Theo 12.)

Is this possible? (See Hyp.)

(1) Therefore is *FS* less than *FV*? (See Ax 20.)

Suppose *FS* greater than *FV*.

Then what is the resulting extensional relation of *OS* and *OV*?

But is this possible?

(2) Therefore is *FS* greater than *FV*?

Therefore what is the relation extensionally, of *FS* to

FV? [Write as authority, the magnitude axiom and the numbers of your statements in answer to questions here numbered (1) and (2).]

I THEOREM 14

Write a theorem specifying the relation of the distances, when the lines drawn as in theorem 13, are unequal. Begin the theorem with the words " Unequal lines from a point in a perpendicular."

Draw the figure making *OV* greater than *OS*, and place the other letters as in the figure for theorem 13. (No development or lettered headings are necessary.)

Demonstrate as follows:

Suppose $FS=FV$. (Insert this statement in your demonstration.)

Then what is the extensional relation of *OV* and *OS*?
Is this possible?
Therefore does $FS=FV$?

Assume *FV* less than *FS*.
Then *OV* is in what relation to *OS*?
But is this possible?
Therefore is *FV* less than *FS*?

What therefore is the relation of *FS* to *FV*?

I THEOREM 15

DEVELOPMENT. Draw two lines perpendicular to the same line.

Will the perpendiculars meet, if produced?

They are therefore in what directional relation to each other?

Demonstration. Write a theorem regarding the directional relation to each other, of lines perpendicular to the same line.

Draw the figure, representing *AB* and *CD*, each perpendicular to *OR*.

Can *AB* meet *CD*? (Hyp and Theo 6.)

Therefore *AB* is in what relation directionally to *CD*?

Axiom 24. ⊥ ity Ax. Draw *AB*, a line of convenient length.

Draw a line not perpendicular to *AB*.

Draw a line perpendicular to *AB*.

Under each line state what the line is.

If possible draw another line in neither of these directional relations to *AB*.

State whether it is possible.

Therefore as regards perpendicularity, lines are in what relations? (Name the relations in the same order as represented in the drawing.)

I THEOREM 16

Development. Draw a line perpendicular to another line *SL*.

Draw a line parallel to the perpendicular, cutting *SL*.

The drawing represents how many parallels, one of which is in what directional relation to *SL*?

Therefore the other is in what directional relation to *SL*?

Demonstration. For the theorem write the answer to the following question:

If one of a number of parallels is perpendicular to a line, the others are in what relation to the line?

Draw the figure, representing *TV* parallel to *SR* which is perpendicular to *AB*.

Suppose *TV* not perpendicular to *AB*.

Through their point of intersection draw *XV* perpendicular to *AB*.

Then *XV* is in what directional relation to *SR*?

But *TV* is in what directional relation to *SR*?

With what therefore does *TV* coincide? (Ax 16.)

But what is the directional relation of *XV* to *AB*?

What therefore is the directional relation of *TV* to *AB*? (Ax 19.)

I THEOREM 17

DEVELOPMENT. Draw a line *AB*.

Draw not less than two lines parallel to *AB*.

Write what the drawing represents.

State the directional relation to each other, of the lines which are parallel to *AB*.

DEMONSTRATION. Lines parallel to the same line are in what directional relation to each other?

Draw the figure representing *AB* and *CD* parallel to *FS*.

By what theorem have lines been proved parallel? *

In order therefore to prove *AB* and *CD* parallel they must be proved to be in what relation to another line? *

Draw *ER* perpendicular to one of three lines in the figure. (Specify which.)

---

* Write the answer to these questions, at the bottom of the page of the work-book.

Therefore each of the other lines is in what directional relation to *ER*? (Answer in two separate statements, naming each line.)

Therefore what is the directional relation of *AB* to *CD*?

I THEOREM 18

DEVELOPMENT. Draw two parallel lines (not less than an inch apart) intersected (obliquely) by another line (a transversal).

State what the drawing represents.

Number each angle formed by the transversal *between* the *parallels,* measure each with the protractor, and enter the measurements.

Which are equal, the non-adjacent angles on the same side of the transversal or the non-adjacent angles on opposite (alternate) sides of the transversal?

What is the name of these non-adjacent angles because of their position *within* the parallels on *alternate* sides of the transversal? (See definitions.)

DEMONSTRATION. If parallels are intersected by a line, what angles are equal? (Use the name given in the definition.)

Draw the figure showing *AB* and *CD* cut by *OV* in the points *T* and *L* respectively. (Draw *OV* oblique and designate one pair of the angles by 1 and 2 and the other pair by 3 and 4.)

Through *S* the middle point of *TL,* draw a line perpendicular to either parallel, cutting *AB* at *E* and *CD* at *F*. (State to what line you draw it perpendicular and therefore its relation to the other parallel.)

Then ∠ *LSF* is in what relation to ∠ *EST*?

On *S* rotate until *SF* falls on *SE*.

Therefore *SL* coincides directionally with what?

But *SL* equals what?

Therefore *L* coincides with what?

Therefore *LF* coincides with what? (Theo 6.)

Therefore 1 is in what relation to 2?
And 3 is in what relation to 4? (Def 22.)

Axiom 25. || ism ax. Letter the heading in the workbook.

Under it draw a line of convenient length.

Under the line write, *AB*, a straight line.

Draw *CD* not parallel to *AB*.

Draw *EF* parallel to *AB*.

Under each of these lines, write what the line is, with reference to *AB*.

If possible, draw a line in neither of these directional relations to *AB*. State whether it is possible.

Therefore as regards parallelism, lines are in what directional relations? (In answering, express the relations in the order in which they were represented, using proper symbols.)

A Converse. A converse is one of two related theorems in which the hypothesis of one is the conclusion of the other, and the conclusion of one is the hypothesis of the other.

For example, the hypothesis of theorem 2, that the adjacent angles have their exterior sides in a straight line, is the conclusion of theorem 3.

The conclusion of theorem 2 that the angles are supplementary is the hypothesis of theorem 3. Each theorem is therefore the converse of the other.

I Theorem 19

Write this theorem as the converse of theorem 18.

Letter the figure as in theorem 18 before construction, and write hypothesis and conclusion.

Assume *CD* not parallel to *AB*. (Ax 25.)

Draw *MN* through *L* parallel to *AB*.

What is the relation of the alternate-interior angles *ATV* and *TLN*?

But by the hypothesis, angle *ATV* equals what angle?

Therefore what is the relation of *TLN* to *TLD*?

Therefore *CD* coincides with what?

But *MN* is in what relation to *AB*?

Therefore *CD* is in what relation to *AB*?

I Theorem 20

Write this theorem, using the same words as in theorem 18, but substituting exterior-interior for alternate-interior.

Base demonstration on theorems 4 and 18. (Do not draw a construction line.)

I Theorem 21

This theorem is the converse of theorem 20. Write it and demonstrate. (Do not draw a construction line.)

I THEOREM 22

DEVELOPMENT. Draw two parallel lines and a transversal.

With the protractor measure two of the interior angles on the *same* side of the transversal.

Their sum equals how many degrees?

DEMONSTRATION. Write a theorem based on this development.

Demonstrate. No construction is necessary.

I THEOREM 23

Write this theorem as the converse of Theorem 22, and demonstrate.

I THEOREM 24

Write this theorem the same as theorem 18, substituting alternate-exterior for alternate-interior.

I THEOREM 25

Write this theorem as the converse of theorem 24, and demonstrate.

I THEOREM 26

DEVELOPMENT. (In this development draw no right angles.)

Near the left margin, draw two angles whose respective sides are parallel and in the same direction from their vertices.

At the right state what the angles are.

Draw two other angles whose respective sides are parallel and extend in opposite directions from their vertices.

At the right state what the angles are.

Draw two more angles with sides respectively parallel, two extending in the same direction and two in opposite directions from their vertices.

State what the angles are.

If possible draw two more angles whose respectively parallel sides extend in none of these directions from their vertices.

State whether it is possible.

Number all of these angles consecutively in the order in which they were drawn.

Measure them with the protractor and enter the measurements.

In which pairs are the angles equal?

In which pair are the angles supplementary?

In which pair are the angles in some other relation?

In every pair of angles, the sides are respectively in what directional relation?

DEMONSTRATION. Write a theorem regarding the relation of angles whose sides are respectively parallel.

Sometimes, though rarely, more than one figure is necessary to represent the different relations expressed in a theorem.

The development work shows such a necessity here, owing to the number of directions in which sides of angles may extend from their vertices.

CASE I. **Sds || and in Same Dir fr Verts.** Enter this heading under the theorem you have written.

Under it draw the figure for this case, and write the hypothesis and the conclusion.

(Be sure not to limit your conclusion by stating only *one* relation. The angles are to be proved equal or supplementary.)

If necessary, produce one or more sides of the angles to intersect.

Demonstration can be effected by the application of some of the theorems on parallel lines and a transversal.

Following your demonstration, write the theorem in the special form represented in case I.

CASE II. **Sds ||, and in Opp Dir fr Verts.**

CASE III. **Sds ||, 2 in same and 2 in Opp Dir fr Verts.**

Write the complete demonstration for each case.

I THEOREM 27

DEVELOPMENT. Near the left margin, at a convenient distance apart, draw two *acute* angles whose sides are respectively perpendicular.

State what the angles are.

Draw two *obtuse* angles whose sides are respectively perpendicular.

State what the angles are.

Draw an acute and an obtuse angle whose sides are in the same specified relation.

If possible draw any other two angles, not right angles and not already named, whose sides are respectively perpendicular.

State whether it is possible.

Number and measure exactly as in the development of theorem 26, and state which pairs are equal, which supplementary, and which in some other relation.

The angles in every pair have their sides respectively in what relation?

DEMONSTRATION. Write a theorem regarding the relations of angles whose sides are respectively perpendicular.

Demonstrate under the following cases.

CASE I. Sds. ⊥, ∠s Acute.
CASE II. Sds. ⊥, ∠s Obtuse.
CASE III. Sds. ⊥, 1 Acute ∠, 1 Obtuse ∠.

Superpose one angle in each pair, so that its vertex is on the vertex of the other, and by construction lines represent its sides in their respective relations of perpendicularity to the sides of the other.

Write the work in the same form as for theorem 26.

BOOK II

TRIANGLES AND PARALLELOGRAMS

1. Sum ∠s △
2. Ext ∠ △

3. Sd and 2 Adj ∠s
4. 2 Sds and Incl ∠

5. Isos △
6. Sds Opp = ∠s

7. 3 Sds =
8. Rt ◺s Sd and Hypot

9. 2 △s 2 Sds =, Incl ∠s ≠
10. 2 △s 2 Sds =, 3rd Sds ≠

11. 1 △, 2 Sds ≠
12. 1 △, 2 ∠s ≠

13. Pt Bis ∠
14. Pt = ly Dist Fr Sds ∠
15. Bis ∠ △s
16. ⊥ Bis Sds △

17. Sum Int ∠s Poly
18. Sum Ext ∠s Poly

19. Quad, Opp Sds =
20. Quad, 2 Sds = and ||

21. Diag ▱
22. Opp Sds ▱
23. Diags ▱
24. 2 Sds and Incl =, ▱s =

25. Series ||s

26. Lines Jng Mid Pts 2 Sds △
27. Mid Sds △ Jnd
28. Line Mid Pts Non || Sds Trap
29. Medians △
30. Diag ▱ Trisected
31. Lines Jng Mid Pts Sds Quad

II THEOREM 1

DEVELOPMENT. Carefully draw a right triangle, number its angles, measure each with the protractor, and write the value of the sum of its angles.

In like manner draw and measure the following triangles, and write the value of the sum of the angles of each: Isoceles, equilateral, acute, obtuse.

Enter all the measurements on the figures.

DEMONSTRATION. Write a theorem regarding the sum of the angles of a triangle.

In order to demonstrate, draw a line through any vertex, parallel to the side opposite that vertex.

Number each of the angles of the triangle and the angles formed at the vertex.

II THEOREM 2

DEVELOPMENT. Draw a triangle with one exterior angle.

What is the relation of the exterior angle to the sum of the two opposite interior angles? (Determine by the protractor.)

Draw at least two other triangles of different shape, and state the relation of an exterior angle of each to the sum of the opposite interior angles. (Determine by the protractor.)

DEMONSTRATION. Write a theorem regarding an * exterior angle of a triangle.

---

* Be sure to write *an* and *not the*. "The" would imply that a triangle can have only one exterior angle.

Write the value of the exterior angle plus its adjacent angle.

Write the value of the sum of the angles of the triangle.

Finish the demonstration.

Method of Indicating Equal Parts. The figures below represent triangles having two sides and the included angle of one, equal respectively to two sides and the included angle of the other.

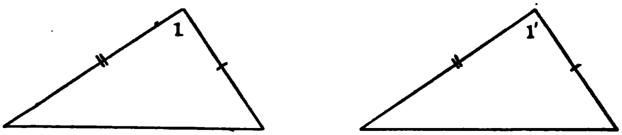


FIG. 1.

The equal angles are shown by the same number, one being primed.

The equal sides are shown by the same number of short lines, drawn perpendicular and very light.

This convention should be observed in the subsequent work.

II THEOREM 3

Triangles are equal if a side and the two adjacent angles of one, equal respectively a side and the two adjacent angles of another.*

Given triangles I and II with AB equal DE, A equal D, and B equal E. (On your figure number the equal angles as above.)

---

* By the two adjacent angles is meant the two angles adjacent to the given side, i.e., the two angles of which the given side is a common side.

Superpose so that *DE* coincides with *AB*.
Then *D* and *E* fall on what points?
And *DF* falls on what?
And *EF* falls on what?
Therefore *F* falls on what? (Ax 14.)
Therefore I coincides with what?
Therefore I is in what relation to II?

II THEOREM 4

Triangles are equal if two sides and the included angle of one equal respectively two sides and the included angle of another.

Superpose so that 1 coincides with 1′ and show where each side and its extremity will fall.

Order of Specification in Superposition. In all superposition specify directional coincidence of lines, before specifying coincidence of points.

II THEOREM 5

DEVELOPMENT. Draw an isosceles triangle. Measure the angles opposite the equal sides and state their relation to each other.

DEMONSTRATION. Write a theorem regarding the relation of the angles opposite the equal sides of an isosceles triangle.

Draw a line from the vertex to the base, bisecting the angle at the vertex.

Write the answers to the first three questions which follow, at the bottom of the page on which you are writing.

What theorems on equal triangles have been proved?

(Answer by writing their numbers and by indicating the theorems in their abbreviated form.)

In the first of these theorems, what parts of the triangles were specified in the hypothesis as equal?

In the second what parts were equal?

Show that the two triangles into which the figure is divided by the construction line from the vertex, have the same parts equal as in *one* of the theorems on equal triangles already proved, and that therefore the triangles are equal.

What is therefore the relation of their homologous angles?

II THEOREM 6

Write this theorem as the converse of theorem 5, beginning with the words, if two angles of a triangle are equal.

Draw a line from the vertex, perpendicular to the base, and demonstrate.

II THEOREM 7

Triangles are equal whose sides are respectively equal.

Superpose one of the triangles so that

(1) one side of it is coincident with the side equal to it in the other triangle;

(2) the other equal sides are adjacent to each other.

Indicate superposed triangle by construction lines.

Connect the opposite vertices and prove equality by showing two sides and the included angles respectively equal.

Importance of the Theorems on Equal Triangles. Theorems 3, 4, and 7 are known as the *general* theorems on equal triangles because applicable to all triangles.

They are so important in the further study of mathematics that they should be memorized and their demonstration studied until thoroughly understood.

In the subsequent work the equality of lines or angles is often proved by showing them to be homologous parts of equal triangles and therefore equal.

How to Prove Triangles Equal. The equality of triangles may be demonstrated by proving the following parts respectively equal:

(1) *One* side and two corresponding angles.
(2) *Two* sides and the included angle.
(3) *Three* sides.

Triangles may also be proved equal by—

(4) Coincidence.

Order of Theorems. The order of the theorems on equal triangles can be easily remembered by observing that in the

First theorem, one side is named;
Second theorem, two sides;
Third theorem, three sides.

Method of Demonstration. Whenever it is necessary to prove triangles equal, they should first be named; then the sides which may be equal should be specified. This requirement is due to the following:

If only *one* side of the one triangle is equal to a side of the other, in order to prove equality it will be necessary to show *two corresponding* angles respectively equal.

If *two* sides only are respectively equal, it will be necessary to prove the *included* angles equal.

If *three* sides are respectively equal, *no* angles will be required.

In other words, the angles and the number of angles which must be proved equal, can be determined only after the equal sides have been named.

Therefore in proving triangles equal, arrange the statements of demonstration in the following order:

(1) Name the triangles.
(2) Specify what sides are equal.
(3) Specify what angles are equal.

II THEOREM 8

Right angles are equal if a side and the hypotenuse of one, equal respectively a side and the hypotenuse of another.

Superpose so that the equal sides (not the hypotenuse) are in coincidence and one triangle does not lie on the other.

Prove that the two sides adjacent to the foot of the perpendicular form a straight line.

Finish the demonstration.

II THEOREM 9

If two sides of a triangle are equal respectively to two sides of another triangle, and the angle included by the two sides of one is greater than the angle included by the two sides of the other, the side opposite the included angle of the one is greater than the side opposite the included angle of the other.

In the triangles *ABC* and *DEF*,* let *AC* equal *DF*, *CB* equal *FE*, and angle *ACB* be greater than angle *DFE*. (For convenience, let *AC* be much shorter than the other sides of *ABC*, and let angle *ACB* be much greater than angle *DFE*.)

---

* In naming a triangle, the first two letters denote the base; the third, the angle of the vertex.

Superpose so that *DF* coincides with *AC*, and indicate by construction lines the position of the other sides of triangle *DFB*.

Draw a line bisecting angle *BCE*, terminating in *AB* at *V*.

Connect *V* and *E*.

Express the relation of $AV+VE$ to AE.

Compare this algebraic expression with the algebraic expression of the conclusion, and determine for what part substitution must be made and what must be substituted in order to obtain the conclusion.

Therefore what must be proved to be the extensional relation between the substituted line and the line for which substitution is made? *

That this relation may exist what must be the relation of the triangles *CVE* and *CVB*.*

Prove this relation, and complete the demonstration.

II THEOREM 10

If two sides of a triangle are equal respectively to two sides of another triangle, but the third side of one is greater than the third side of the other, the angle opposite the third side of the one is greater than the angle opposite the third side of the other.

In what relation is this theorem to theorem 9?*

What relations only, may two angles sustain to each other? *

---

* Write answer at the bottom of the page of the work-book.

Demonstrate by assuming successively that each of these relations exists, and state in what way each of these assumptions affects the relations of the two opposite sides of the triangles.

State finally what relation of the two angles, only is possible.

(If unable to determine the form of demonstration, refer to I theorem 13.)

II THEOREM 11

DEVELOPMENT. Draw a triangle with two sides unequal.

Measure the angles opposite the unequal sides.

Which is greater, the angle opposite the long side or the angle opposite the short side?

DEMONSTRATION. Write a theorem from the development.

Show triangle ABC with $BC > CA$.

On BC, from C mark off CV equal to CA.

Draw AV.

Express the relation of the angle CVA to angle B plus angle BAV. (Designate the angles by number.)

What is the relation therefore, of angle CVA to angle B?

To what other angle is CVA equal?

Substitute and complete the demonstration.

II THEOREM 12

Write this theorem as the converse of theorem 11.

INDIRECT METHOD OF DEMONSTRATION. See II Theorem 10 for form of demonstration.

DIRECT METHOD OF DEMONSTRATION. Lay off the small angle on the large angle by drawing a line from the vertex of the large angle to the opposite side.

II THEOREM 13

Every point in the bisector of an angle is equally distant from the sides of the angle.

Represent by solid lines the distances* specified, and demonstrate.

II THEOREM 14

Write this theorem as the converse of theorem 13 and demonstrate.

II THEOREM 15

The bisectors of the angles of a triangle, produced if necessary, meet in a point.

Draw the bisectors of the angles much lighter than the sides of the triangles.

Prove first, that any two of the bisectors meet, say the bisectors AX and BY, as follows:

In order to show that two lines meet, what directional relation between the two lines must be proved to be impossible?

If AX and BY do not meet at some point as O, to what number of degrees must the sum of the angles OBA and OAB be not equal?

Prove their sum not equal to this number of degrees by proving A plus B not equal to it, and therefore the sum of a part of A and a part of B (OBA plus OAB) not equal to it, writing work in full.

---

* See note on page 33, regarding distance.

In what extensional relation is O's distance from AB to its distance from BC?

What is the extensional relation of its distance from AB and AC? (Express answer to each question as an equation.)

PROBLEM IN CONSTRUCTION. To inscribe a circle in a triangle.

Draw a triangle and inscribe a circle, obtaining center and radius by II Theorem 15.

II THEOREM 16

The perpendicular bisectors of the sides of a triangle, produced if necessary in one direction or the other, meet in a point.

Draw different kinds of triangles until you have determined under how many cases it is necessary to treat this theorem; then draw the triangles which represent the case or cases, numbering them as Case I, etc., and demonstrate.

PROBLEM IN CONSTRUCTION. To circumscribe a circle about a triangle. Draw a triangle and circumscribe a circle, obtaining center and radius by II theorem 16.

II THEOREM 17

DEVELOPMENT. Draw a polygon having five sides, all unequal.

Measure its angles and state what their sum equals.

Draw a polygon of the same number of sides, but all equal, and find the value of the sum of the angles.

Do likewise with polygons having six, seven, and ten sides.

Number each polygon in succession in Roman notation.
Summarize results as follows:

In I, sum ∠s = how many St ∠s?
How many less in number than the number of sides?
In II, (Answer the same questions).
In III, (Answer the same questions).
Therefore in a polygon of *n sides*, the sum of the angles equals (*n*−?) St ∠s?

DEMONSTRATION. Write the theorem, join any point within the polygon to each of the vertices, and demonstrate.

II THEOREM 18

DEVELOPMENT. Draw a polygon having five sides, produced in succession at one extremity only.

Number and measure the exterior angles thus formed at each vertex.

The sum of the exterior angles equals how many straight angles?

Find whether the sum of the exterior angles has the same value when a polygon has a greater number of sides. (Show all work.)

Do the results indicate that the sum of the exterior angles depends or does not depend, on the number of sides?

DEMONSTRATION. The sum of the exterior angles of a polygon formed by producing its sides in succession, equals how many straight angles?

In writing the hypothesis, denote the number of vertices by *n*.

Demonstrate in the following form:

Int ∠ + Ext ∠ at each Vertex = how many St ∠s?

But there are how many vertices?

∴ Sum Int ∠s + Sum Ext ∠s = ?

But Sum Int ∠s = ?

Finish the demonstration, showing all work in the second member of the final equation.

II THEOREM 19

DEVELOPMENT. Draw a quadrilateral whose *opposite* sides are equal. (Do not make all sides equal, and do not draw a rectangle.)

Find the value of the sum of the interior angles formed by two of the opposite sides with one of the other sides.

The two opposite sides are therefore in what directional relation to each other.

In the same way ascertain the directional relation of the other pair of opposite sides.

The quadrilateral is therefore what kind of a figure? (See definitions.)

DEMONSTRATION. Write a theorem regarding a quadrilateral whose opposite sides are equal, and demonstrate.

II THEOREM 20

DEVELOPMENT. Determine what kind of figure is formed by a quadrilateral having two sides equal and parallel. (Show all work.)

DEMONSTRATION. Write the theorem and demonstrate.

Number each polygon in succession in Roman notation. Summarize results as follows:

In I, sum $\angle$s = how many St $\angle$s?

How many less in number than the number of sides?

In II, (Answer the same questions).

In III, (Answer the same questions).

Therefore in a polygon of n *sides*, the sum of the angles equals ($n-$?) St $\angle$s?

DEMONSTRATION. Write the theorem, join any point within the polygon to each of the vertices, and demonstrate.

II THEOREM 18

DEVELOPMENT. Draw a polygon having five sides, produced in succession at one extremity only.

Number and measure the exterior angles thus formed at each vertex.

The sum of the exterior angles equals how many straight angles?

Find whether the sum of the exterior angles has the same value when a polygon has a greater number of sides. (Show all work.)

Do the results indicate that the sum of the exterior angles depends or does not depend, on the number of sides?

DEMONSTRATION. The sum of the exterior angles of a polygon formed by producing its sides in succession, equals how many straight angles?

In writing the hypothesis, denote the number of vertices by n.

Demonstrate in the following form:

Int ∠ + Ext ∠ at each Vertex = how many St ∠s?

But there are how many vertices?

∴ Sum Int ∠s + Sum Ext ∠s = ?

But Sum Int ∠s = ?

Finish the demonstration, showing all work in the second member of the final equation.

II THEOREM 19

DEVELOPMENT. Draw a quadrilateral whose *opposite* sides are equal. (Do not make all sides equal, and do not draw a rectangle.)

Find the value of the sum of the interior angles formed by two of the opposite sides with one of the other sides.

The two opposite sides are therefore in what directional relation to each other.

In the same way ascertain the directional relation of the other pair of opposite sides.

The quadrilateral is therefore what kind of a figure? (See definitions.)

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DEMONSTRATION. Write the theorem and demonstrate.

Number each polygon in succession in Roman notation.

Summarize results as follows:

In I, sum $\angle$s = how many St $\angle$s?

How many less in number than the number of sides?

In II, (Answer the same questions).

In III, (Answer the same questions).

Therefore in a polygon of *n sides,* the sum of the angles equals $(n-?)$ St $\angle$s?

Demonstration. Write the theorem, join any point within the polygon to each of the vertices, and demonstrate.

II Theorem 18

Development. Draw a polygon having five sides, produced in succession at one extremity only.

Number and measure the exterior angles thus formed at each vertex.

The sum of the exterior angles equals how many straight angles?

Find whether the sum of the exterior angles has the same value when a polygon has a greater number of sides. (Show all work.)

Do the results indicate that the sum of the exterior angles depends or does not depend, on the number of sides?

Demonstration. The sum of the exterior angles of a polygon formed by producing its sides in succession, equals how many straight angles?

In writing the hypothesis, denote the number of vertices by n.

Demonstrate in the following form:

Int ∠ + Ext ∠ at each Vertex = how many St ∠s?

But there are how many vertices?

∴ Sum Int ∠s + Sum Ext ∠s = ?

But Sum Int ∠s = ?

Finish the demonstration, showing all work in the second member of the final equation.

II THEOREM 19

DEVELOPMENT. Draw a quadrilateral whose *opposite* sides are equal. (Do not make all sides equal, and do not draw a rectangle.)

Find the value of the sum of the interior angles formed by two of the opposite sides with one of the other sides.

The two opposite sides are therefore in what directional relation to each other.

In the same way ascertain the directional relation of the other pair of opposite sides.

The quadrilateral is therefore what kind of a figure? (See definitions.)

DEMONSTRATION. Write a theorem regarding a quadrilateral whose opposite sides are equal, and demonstrate.

II THEOREM 20

DEVELOPMENT. Determine what kind of figure is formed by a quadrilateral having two sides equal and parallel. (Show all work.)

DEMONSTRATION. Write the theorem and demonstrate.

Number each polygon in succession in Roman notation. Summarize results as follows:

In I, sum $\angle$s = how many St $\angle$s?

How many less in number than the number of sides?

In II, (Answer the same questions).

In III, (Answer the same questions).

Therefore in a polygon of *n sides*, the sum of the angles equals $(n-?)$ St $\angle$s?

DEMONSTRATION. Write the theorem, join any point within the polygon to each of the vertices, and demonstrate.

II THEOREM 18

DEVELOPMENT. Draw a polygon having five sides, produced in succession at one extremity only.

Number and measure the exterior angles thus formed at each vertex.

The sum of the exterior angles equals how many straight angles?

Find whether the sum of the exterior angles has the same value when a polygon has a greater number of sides. (Show all work.)

Do the results indicate that the sum of the exterior angles depends or does not depend, on the number of sides?

DEMONSTRATION. The sum of the exterior angles of a polygon formed by producing its sides in succession, equals how many straight angles?

In writing the hypothesis, denote the number of vertices by *n*.

Demonstrate in the following form:

Int ∠ + Ext ∠ at each Vertex = how many St ∠s?

But there are how many vertices?

∴ Sum Int ∠s + Sum Ext ∠s = ?

But Sum Int ∠s = ?

Finish the demonstration, showing all work in the second member of the final equation.

II THEOREM 19

DEVELOPMENT. Draw a quadrilateral whose *opposite* sides are equal. (Do not make all sides equal, and do not draw a rectangle.)

Find the value of the sum of the interior angles formed by two of the opposite sides with one of the other sides.

The two opposite sides are therefore in what directional relation to each other.

In the same way ascertain the directional relation of the other pair of opposite sides.

The quadrilateral is therefore what kind of a figure? (See definitions.)

DEMONSTRATION. Write a theorem regarding a quadrilateral whose opposite sides are equal, and demonstrate.

II THEOREM 20

DEVELOPMENT. Determine what kind of figure is formed by a quadrilateral having two sides equal and parallel. (Show all work.)

DEMONSTRATION. Write the theorem and demonstrate.

II THEOREM 21

DEVELOPMENT. Draw a parallelogram having one diagonal.

Measure the parts which may be necessary to show by three different theorems, the relation of the triangles into which the parallelogram is divided.

DEMONSTRATION. Write a theorem regarding a diagonal of a parallelogram, and demonstrate.

II THEOREM 22

Write a theorem regarding the extensional relation of the opposite sides of a parallelogram, and demonstrate.

II THEOREM 23

DEVELOPMENT. Determine the extensional relation of the parts into which the diagonals of a parallelogram cut each other.

DEMONSTRATION. Write the theorem and demonstrate.

II THEOREM 24

Parallelograms are equal if two sides and the included angle of one equal respectively two sides and the included angle of another.

Draw a diagonal in each, not dividing the given equal angles, and prove by equal triangles.

II THEOREM 25

DEVELOPMENT. Near the left margin draw a line about three inches long, somewhat oblique.

At the right draw another oblique, converging line, about midway between margins.

With dividers step off four equal distances on the first line.

Through the points of division, draw parallels cutting both lines.

Measure the segments intercepted on the second oblique line, and state their relation to each other.

DEMONSTRATION. Parallels intercepting equal segments on one transversal, intercept what segments on any other transversal?

From the respective points of intersection of one of the transversals with two of the parallels, draw lines parallel to the other transversal, terminating each line in the adjacent parallel.

Prove the equality of the two triangles thus formed.

Complete the demonstration.

II THEOREM 26

DEVELOPMENT. Draw a line joining the middle points of two sides of a triangle, terminating in the sides.

Find the value of the sum of the interior angles formed by the line, the third side of the triangle, and one of the bisected sides.

The line is therefore in what directional relation to the third side?

Determine its extensional relation to the third side.

Demonstration. Write the theorem specifying the relations of the line, in the order of development.

To prove the first relation, assume non-parallelism and through either point of bisection draw a line parallel to the third side.

Show coincidence of the construction line and the given line.

To prove the other relation, from one extremity of the given line draw a line parallel to the side of the triangle in which the other extremity terminates, and prove by equality of triangles.

II Theorem 27

Development. Represent a triangle having the middle points of its three (unequal) sides joined in succession.

Do the four triangles into which the triangle is divided seem to be equal or unequal?

If they are equal, what is the relation of their sides respectively by a previous theorem?

Measure and state result.

Demonstration. A triangle having the middle points of its sides joined in succession, is divided into triangles in what relation to each other?

II Theorem 28

A line bisects the non-parallel sides of a trapezoid, terminating in these sides.

Ascertain its directional relation to the parallel sides, its extensional relation to their sum, write the theorem, and demonstrate, entering your work under proper headings.

II THEOREM 29

DEVELOPMENT. Draw a triangle with its three medians.

If accurately drawn the medians will pass through a common point.

Determine the distance of the point from each side of the triangle measured on the median.

The distance from each side, on the median, equals what part of the length of the median to the side?

DEMONSTRATION. The medians of a triangle pass through a common point whose distance from any side, measured on the median, equals what part of the median to that side?

Show that the medians AX and BY intersect at some point, as O.

Join middle points of AO and BO, connect X with Y, connect X with middle point of BO, and Y with middle point of AO.

Prove that the figure formed by the construction lines is a parallelogram, and complete the demonstration for the medians AX and BY.

Assume that BY and CS do not intersect at O but at O'.

Join middle points of CO' and BO', connect Y with S, Y with middle point of CO', S with middle point of BO' and proceed as in the case of AX and BY.

Finally, show that O' coincides with O.

II THEOREM 30

DEVELOPMENT. Represent the middle points of two opposite sides of a parallelogram, each joined to the nearer opposite vertex; also show a diagonal drawn from the other vertices.

Determine the extensional relation of the three segments of the diagonal.

In another figure represent the middle points joined each to the further opposite vertices.

Determine the relation of the segments of the diagonal.

Ascertain if the same relation exists when the middle points of the other opposite sides are joined to the nearer or the further opposite vertex.

Therefore in every instance the diagonal is how many sected?

DEMONSTRATION. Write the theorem.

To demonstrate, draw a line through the extremities of the diagonal, parallel to the lines from the middle points.

II THEOREM 31

DEVELOPMENT. Draw a large quadrilateral whose sides differ considerably in length.

Bisect the sides and join the middle points in succession by heavy lines.

In three different ways* ascertain the kind of figure which is formed by the heavy lines.

Determine if the figure is of the same kind when the quadrilateral is of different shape.

DEMONSTRATION. Lines joining in succession the middle points of the sides of a quadrilateral form what kind of a figure?

Demonstrate.

NOTE. If time allows the exercises and supplementary theorems based on books I and II, should now be taken.

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* Enter all work.

BOOK III

CIRCLES

1. Diam >
2. Diam Bis
3. Chds =, ∠s =
4. ∠s =, Chds =
5. ∠s =, ⌒s =
6. ⌒s =, ∠s =
7. > ⌒, > ∠
8. > ⌒, > chd
9. > Chd, > ⌒
10. Rad ⊥ Chd, Bis
11. Rad Bis Chd, ⊥
12. ⊥ Bis Chd
13. Chds =, Dists =
14. Dists =, Chds =
15. > Chd Nearer
16. Nearer Chd, >
17. ⊥ Ext Rad, Tan
18. Tan, ⊥ Rad
19. 1 Circ Thru 3 Pts
20. ||s Inters = ⌒s
21. Tans Fr Pt Without
22. Line Centers ⊥
23. Line Centers Thru Pt Tan
24. Rat Cent ∠s
25. Circum = 360°
26. Cent ∠ ⊐
27. Inscr ∠ ⊐
28. ∠ Tan and Chd
29. ∠ 2 Chds
30. ∠ 2 Secs

III THEOREM 1

Write a theorem regarding the extensional relation between the diameter of a circle and any other chord.

To demonstrate, make the chord and the diameter coincident at one extremity, denoting new position of chord by a construction line. Connect the other extremity of the construction chord with the center of the circle.

Demonstrate also by connecting with the center, the extremities of the chord when not coincident at either extremity with the diameter.

III THEOREM 2

Write a theorem regarding the relation of the parts into which a circle and the circumference are divided by a diameter.

Demonstrate by folding on the diameter as an axis.

III THEOREM 3

DEVELOPMENT. Draw two equal circles.

Draw a chord in one equal to a chord in the other.

Ascertain the relation of the central angles subtended by the equal chords.

DEMONSTRATION. Equal chords of the same circle or of equal circles, subtend central angles in what relation to each other?

Demonstrate.

III THEOREM 4

Write this theorem as the converse of theorem 3, and demonstrate.

III THEOREM 5

Write a theorem regarding the relation of arcs of the same circle or of equal circles, intercepted by equal central angles.

Demonstrate by superposition.

III THEOREM 6

Write this theorem as the converse of theorem 5.

III THEOREM 7

In the same circle or in equal circles, does the greater of two unequal arcs subtend the greater, or the smaller, central angle?

Construct as follows:

From one extremity of the greater arc, lay off the smaller on the greater, and draw radii.

III THEOREM 8

Write this theorem in the same form as theorem 7, specifying chords instead of angles.

Draw radii from the extremities of the chords.

In the two triangles formed by these radii, what parts are equal?

What parts unequal?

III Theorem 9

Write this theorem as the converse of theorem 8.

Make the chords coincident at one extremity, placing the arc subtended by one, on the subtended arc of the other.

Draw radii.

In the triangles thus formed, what parts are equal?

III Theorem 10

Development. Draw a radius perpendicular to a chord.

Determine the relation of the segments of the chord, and of the segments of the subtended arc.

Demonstration. Write the theorem and demonstrate.

III Theorem 11

Development. Draw a radius bisecting a chord.

Determine its directional relation to the chord.

Demonstration. Write the theorem and demonstrate.

III Theorem 12

A perpendicular bisector of a chord, produced if necessary, passes through the center of the circle.

Assuming that the perpendicular does not pass through the center of the circle, prove that it coincides with a radius drawn through the center of the chord.

Problem. Find the center of a circle.

III THEOREM 13

DEVELOPMENT. Draw a circle and two equal chords.

Draw lines representing the chords' distances from the center.

Determine the relation of the distances.

DEMONSTRATION. Write the theorem and demonstrate.

(Draw a radius to one extremity of each chord.)

III THEOREM 14

Write and demonstrate as the converse of theorem 13.

III THEOREM 15

The greater of two unequal chords is at the greater distance from the center of the circle.

Make the chords coincident at one extremity, with the arc subtended by one *not* lying on the arc subtended by the other.

Draw lines representing the distances of all the chords from the center of the circle.

Connect the centers of the chords now coincident at one extremity.

Consider first, the triangle having the connecting line as one of its sides and whose other two sides are the segments of the chords which are coincident at one extremity.

Prove two sides of this triangle unequal.

State the relation of the angles opposite these sides.

Therefore prove the relation of the angles adjacent to these in the other triangle, and finish the demonstration.

III THEOREM 16

Write this theorem as the converse of theorem 15.

Demonstrate either by the indirect method or in the same way as the preceding theorem.

III THEOREM 17

A line perpendicular to a radius at its extremity is tangent to the circle.

Draw any other line from the center of the circle to the given line and prove the radius the shorter.

III THEOREM 18

Write this theorem as the converse of theorem 17, and demonstrate.

III THEOREM 19

A circumference and one only, can be passed through three points not in a straight line.

Connect the two outer points with the third.

Erect perpendiculars at the middle of each of the connecting lines.

III THEOREM 20

Write a theorem regarding the relation of arcs intercepted by parallels.

For the figure draw three circles, the first with two parallel tangents; the second with two parallel secants;

and the third with a secant and a tangent which are parallel.

Mark the different circles successively as CASE I, CASE II, etc.

In CASE I draw radii to the points of tangency.

By the hypothesis and the theorem regarding the number of perpendiculars from a point without a line, prove that these two radii form a diameter.

In CASE II draw a diameter perpendicular to one of the parallels.

In CASE III draw a radius or a diameter to the point of tangency.

III THEOREM 21

DEVELOPMENT. Draw a circle and locate a point without the circumference.

From the point draw two tangents which terminate in the points of tangency.

Determine the extensional relation of the tangents to each other, and the relation of the angles formed by the tangents and a line joining the given point with the center of the circle.

DEMONSTRATION. Write a theorem specifying these relations.

To demonstrate, draw radii to the points of tangency.

III THEOREM 22

DEVELOPMENT. Draw two intersecting (unequal) circles.

Draw the line of centers.

Draw their common chord. (A common chord of two circles is a line joining their points of intersection.)

Determine and state the directional relation of the line of centers to their common chord; also the extensional relation of the segments of the chord.

Demonstration. Write a theorem regarding the directional relation of the line of centers of two intersecting circles, to their common chord, and the extensional relation of the segments of the chord, to each other.

In demonstrating assume non-perpendicularity.

Draw a line perpendicular to the common chord at its middle point.

Produce this line at each extremity if necessary, and prove it coincident with the line of centers.

III Theorem 23

The line of centers of two tangent circles passes through the point of tangency.

Case I. Tangent externally.

Case II. Tangent internally.

Assuming that the line of centers does not pass through the point of tangency, draw a perpendicular to the common tangent at the point of tangency.

Prove that this perpendicular passes through the centers.

III Theorem 24

Angles at the center of equal circles or of the same circle, are in the same ratio as their intercepted arcs, i.e., the ratio of the angles equals the ratio of their respective arcs.

Case I. **The arcs commensurable.**

Draw two equal circles, each having an angle at the center, O and O', intercepting the commensurable arcs AR and VS.

State the hypothesis and the conclusion, expressing the conclusion as an equation.

Demonstrate as follows:

AR and VS are what kind of quantities?

Therefore they may be divided by what kind of measure?

Lay off this measure m, (specify what kind) on each, as many times as contained, in AR s times, and in VS r times.

Then AR equals how many times m?

And VS equals how many times m?

Therefore AR/VS equals what?

Join the points of division of each arc with the center.

Then the angle O is divided into how many equal parts?

And the angle O' is divided into how many equal parts?

Therefore O/O' equals what?

Compare the ratio with the one previously obtained and finish the demonstration.

CASE II. **The arcs incommensurable.** (This demonstration will be considered in Book VII.)

Some Important Definitions

A Degree. In definition 15, a degree is defined as an angle equal to one-ninetieth of a right angle. This conception of a degree as an angle and therefore as a plane figure formed by two straight lines diverging from a common point, is generally not understood. Probably the majority of students of geometry, even on completion of the subject, have no idea of a degree beyond the fact that there are 360 degrees in a circumference, a fact which only a clear understanding of the meaning both of a degree and an arc degree can make intelligible.

Definition of an Arc Degree. An arc degree is a segment of the circumference intercepted by a central angle of one degree.

Except in the definitions, the characterizing words arc and angle are unnecessary, for the reason that when a degree is named with reference to angles, an angle degree is understood; when named with respect to a circumference, an arc degree is understood.

Measurement of Angles. By definition 62, an angle measured by its intercepted arc is an angle whose number of angle degrees equals the number of intercepted arc degrees.

Therefore an angle measured by *half* its intercepted arc is an angle whose number of angle degrees equals one-half the number of intercepted arc degrees.

III Theorem 25

A circumference equals 360°.

Draw a diameter.

III Theorem 26

A central angle is measured by its intercepted arc.

In the hypothesis, name the given angle as an angle of $n°$.

Demonstrate as follows:

Cent ∠ 1 equals how many degrees?
Therefore ⌒ AB equals how many degrees?
(=Cent ∠s Interc = ⌒s and Def ⌒ °).
Therefore Cent ∠ 1 is measured by what?
(See Def ∠ ⊐ ⌒).

THE PROTRACTOR. Following your demonstration, explain the application of this theorem to the protractor and its use.

III THEOREM 27

An inscribed angle is measured by one-half the intercepted arc.

In a circle inscribe an angle, one of whose sides passes through the center.

In another circle inscribe an angle having the center of the circle between its sides.

In another circle inscribe an angle with its sides in neither of these relations to the center.

Number the different cases consecutively in Roman notation.

Demonstrate Case I, by connecting one extremity of the intercepted arc with the center.

III THEOREM 28

An angle formed by a tangent and a chord, is measured by one-half the intercepted arc.

If the chord is not a diameter, draw a diameter from the point of tangency.

III THEOREM 29

An angle formed by two chords which intersect within the circumference, is measured by one-half the sum of the arcs intercepted by the angle and its opposite angle.

From one extremity of either chord, draw a chord parallel to the other chord; or, connect the chords at either extremity.

III THEOREM 30

An angle formed without a circumference, by two secants, two tangents, or a tangent and a secant, is measured by one-half the differance of the arcs included between the points of contact or intersection of the lines with the circumference.

Use three circles.

In Cases I and III, connect two opposite and non-adjacent points of intersection of the lines with the circumference; or from the point of intersection of one of the lines with the circumference nearer the external point, draw a chord parallel to the other line forming the external angle.

In Case II, connect the points of tangency; or, from either point of tangency draw a chord parallel to the other tangent.

> "We fayle of that which we wolden have,
> And in our madnesse evermore we rave.
> And whan we been togidres everichoon,
> Every man semeth a Salomon.
> But al thing which that shyneth as the gold
> Nis nat gold, as I have herd it told;
> Ne every appel that is fair at yē
> Ne is nat good, what-so men clappe or crye.
> Right so, lo! fareth it amonges us;
> He that semeth the wysest,
> Is most fool whan it cometh to the preef."—CHAUCER.

BOOK IV

PROPORTION

1. Prod Means =
2. Mean Proportional
3. 2 Prods 2 Factors Ea
4. Alternation
5. Inversion
6. Composition
7. Division
8. Comp and Div
9. Equimultiples
10. Prod Cor Terms
11. Like Powers and Roots
12. Sum Antecedents

Remark. The most difficult parts of Euclid's Elements, were the books on proportion because the theorems were proved by geometric methods. The algebraic method makes proportion one of the simplest topics in mathematics and it is therefore used in all modern books on geometry. Although the algebraic treatment makes proportion no longer a geometric subject, it is still retained in the geometries because of its application to similar triangles.

IV THEOREM 1

In every proportion the product of the means equals the product of the extremes.

Given $\frac{a}{b}=\frac{c}{d}$

Prove $bc=ad$

IV THEOREM 2

A mean proportional between two quantities equals the square root of their product.

Given $\frac{a}{b}=\frac{b}{c}$

State the conclusion and demonstrate.

IV THEOREM 3

If two products of two factors each are equal, the factors of either product may be made the extremes of a proportion in which the factors of the other product are made the means.

Given $ad=bc$

Express the conclusion as an equation.

IV THEOREM 4

The terms of a proportion will be in proportion by alternation.

Given $\frac{a}{b}=\frac{c}{d}$

Express the conclusion as an equation.

Compare the first fraction of the hypothesis with the first fraction of the conclusion and determine by what one must be multiplied or divided in order to transform it into the other.

IV THEOREM 5

The terms of any proportion will be in proportion by inversion.

Write the same proportion as in theorem 4.

In what operation is a quantity inverted and used as a multiplier?

Therefore divide 1 by the equation in which the hypothesis is expressed.

IV THEOREM 6

The terms of any proportion will be in proportion by composition.

Add 1 to each member of the equation in which the hypothesis is expressed.

Simplify by adding the terms in each member. (The authority is the fraction axiom.)

IV THEOREM 7

The terms of any proportion will be in proportion by division.

IV THEOREM 8

The terms of any proportion will be in proportion by composition and division.

IV THEOREM 9

Equimultiples of two or more quantities are in the same ratio as the given quantities.

Given $\frac{a}{b}=r$

Prove $\frac{ma}{mb}=r$

IV THEOREM 10

The products of the corresponding terms of two or more proportions are in proportion.

Express at least three proportions algebraically.

IV THEOREM 11

Like powers and like roots of the terms of a proportion are in proportion.

IV THEOREM 12

In a series of equal ratios, the ratio of the sum of the antecedents to the sum of the consequents, equals the ratio of any antecedent to its consequent.

Write a series of three equal literal ratios.

Since the ratios are all equal, they will equal the same quantity.

Express this in a series of equations, making each ratio equal to s.

Clear each equation of fractions, add the resulting equations, and solve for the literal quantity to which each ratio was made equal.

BOOK V

SIMILAR TRIANGLES

1. Line Thru 2 Sds || 3rd
2. Segs Pro, Line ||
3. Mut = ∠ular △s
4. Cor Sds Pro
5. I Pr ∠s =, Incl Sds Pro
6. Sds Resp || or ⊥
7. Cor Alts ∼ △s
8. Cor Intercs || s
9. Transv Mkg Pro Intercs
10. Bis ∠ △
11. Bis Ext ∠ △
12. ⊥ to Hypot
    (1) ∼ △s
    (2) ⊥ M P Segs Hyp
    (3) Ea Leg M P Hyp and Adj Seg
13. ⊥ to Diam
14. Square Hypot

PROB. 3. Determine Rad fr Chd and Rise

15. Sq Sd Opp Acute ∠
16. Sq Sd Opp Obtuse ∠
17. Sum Sqs Sds △
18. Diff Sqs Sds △
19. Median △
20. Prod 2 Sds
21. Prod Segs Chd
22. Sum Sqs Sds ▱
23. Sq Bis ∠ △
24. Prod Sec and Ext Seg
25. Tan M P Sec and Ext Seg

PROB. 4. Divide Line in Ext and Mean Rat

26. Line Jng Center, Bis
27. Alt △
28. Bis ∠s △ Terms Sd
29. Sum Sqs Sds Quad

Definition of Similar Triangles

Development. In the work which follows the greatest possible accuracy is necessary.

Enter the heading in the work-book.

Near the left margin draw a triangle of any convenient size.

With a protractor measure its angles, entering the measurements.

To the right draw a second triangle of exactly the same shape as the first, but much larger.

In order to make the second triangle of the same shape as the first, it was necessary to make what parts equal to those of the first?

If two triangles are of the same shape, therefore, the angles of one are in what relation to the angles of the other?

Triangles of the same shape are called similar.

The two triangles are therefore in what relation?

What parts do they have respectively equal?

If asked to write a definition of similar triangles you would therefore specify that they are triangles having what parts respectively equal?

Under your answer to the last question write a heading

Partial Definition of Similar Triangles.

Under this write the definition as now developed.

In order to determine the second part of the definition measure with the scale any side of the first triangle.

Measure the side opposite the equal angle of the second triangle.

The ratio of the two sides equals what fraction? (See definition ratio.)

Measure a second side of the first triangle.

Measure the side opposite the equal angle in the second triangle.

To what fraction is the ratio of these two sides equal?

Measure the remaining sides and write their ratio in like manner.

Reduce each ratio, carrying the results to three places.

What therefore is the relation of the three ratios to each other? (Express as a continued proportion. See definition 77.)

Therefore similar triangles have their homologous sides in what relation?

Your answer if correct is the second part of the definition of similar triangles.

Under a heading Definition of Similar Triangles, write the complete definition as now determined.

From this definition it follows that when triangles are similar, their corresponding angles are in what relation?

And their homologous sides are in what relation?

V THEOREM 1

If two of the sides of a triangle are cut by a line parallel to the third side, the ratio of the segments of one side equals the ratio of the corresponding segments of the other side.

CASE I. The segments of one side commensurable.

Draw a triangle *ABC*, with *RS* cutting *AC* and *CB* in points *R* and *S* respectively, making *AR* and *RC* commensurable.

AR and *RC* are what kind of quantities?

Therefore they may be evenly divided by what kind of a measure?

Lay off this measure *m*, (specify what kind) on each of the segments from *R*, as many times as contained, *r* times in *AR* and *s* times in *RC*.

Then AR = how many times m?
And RC = how many times m?
Therefore AR/RC equals what?

Given two transversals AC and CB, one of which is divided into equal parts; in what directional relation to one another must lines be drawn through the points of division of AC in order to intercept equal segments on CB?*

Write directions for drawing these lines and state the number of equal parts into which they divide BS and SC.
Therefore BS/SC equals what?
Compare this ratio with the one previously obtained and finish the demonstration.

CASE II. **Segments Incommensurable.** (Demonstration of this case is presented in Book VII.)

V THEOREM 2

A line dividing two of the sides of a triangle into proportional segments is parallel to the third side.

State the hypothesis as an equation.

Assume the line not parallel to the third side.
Through the point of intersection of the line with *one* of the sides, pass a line parallel to the third side, intersecting the other side.
State the relation of the corresponding segments into which the parallel cuts the two sides.
To this algebraic expression and to the algebraic expression of the hypothesis, apply axiom 1, and transform the resulting proportion by composition.
Add the terms of the numerators, solve for the denominator of the first fraction, and complete the demonstration.

* Write answer at the bottom of the page.

V THEOREM 3

Mutually equiangular triangles are similar.

Draw the figure, making one triangle considerably larger than the other.

Refer to the definition of similar triangles in order to determine what relation of parts in addition to the relation specified in the theorem, is essential to similarity.

In order to show this relation superpose the smaller triangle on the greater so that two equal angles coincide, representing by a construction line the position of the third side of the superposed triangle.

Prove the directional relation of the construction side to the corresponding side of the greater triangle.

Therefore express the relation of the segments into which the construction line cuts two of the sides of the greater triangle, and apply IV theorem 6.

In order to prove the necessary relation of the remaining parts, superpose so that another pair of equal angles coincide, and demonstrate in like manner.

V THEOREM 4

Triangles whose corresponding sides are proportional, are similar.

By reference to the definition of similar triangles determine what element of similarity is lacking in the triangles specified in the theorem.

On two of the sides of the greater triangle lay off from their vertex the two corresponding sides of the smaller triangle.

Connect the extremities of the sides laid off.

From the hypothesis, the construction, and IV theorem 7, in what directional relation is the connecting line to the third side of the triangle, two of whose sides it connects?

Therefore prove the construction triangle and the greater triangle similar.

Form a proportion (I), whose first term is the construction side.

It will now be necessary to prove the construction triangle equal to the smaller given triangle.

Therefore name them and specify the essentials of equality known to exist between them:

(1) their equal sides,

(2) their equal angles.

What relation between their third sides must therefore be proved?

In order to prove this relation, form a proportion (II), whose first term is a side of the smaller given triangle corresponding to the construction side of the other triangle, two of whose other sides are terms of proportion (I).

Prove the equality of the first terms of proportion (I) and (II).

What therefore is the relation of the smaller given triangle to the construction triangle?

Therefore what is the relation of their homologous angles?

But the angles of the construction triangle are in what relation to the angles of the greater triangle?

Finish the demonstration.

V THEOREM 5

Triangles which have an angle of one equal to an angle of the other and the including sides proportional are similar.

By what theorems have triangles been proved similar?

Superpose the smaller triangle on the greater so that the equal angles coincide, representing the third side by a construction line.

Prove similarity by one of the theorems named.

V THEOREM 6

Triangles whose sides are respectively parallel or perpendicular are similar.

Draw figures to represent both parallelism and perpendicularity, numbering them as Case I and Case II, respectively.

In the demonstration of the preceding theorem what essentials of similarity were proved?

Because of these essentials the given triangles were therefore in what relation?

Prove the same essentials in each case of this theorem.

Essentials of Similarity. 1. Corresponding angles equal and homologous sides proportional. (Definition.)

Similar triangles therefore have the same shape but are *not* necessarily of the same size.

2. Mutually equiangular. (Theorem 3.)

3. Homologous sides proportional. (Theorem 4.)

4. One angle of one equal to an angle of the other and the including sides respectively proportional. (Theorem 5.)

5. Sides respectively parallel or perpendicular. (Theorem 6.)

The Use of Similar Triangles. The theorems on similar triangles are so important that their mastery is a necessity. Attention is therefore called to the following:

By the first specification of the definition of similar triangles, they are mutually equiangular; by the second, their corresponding sides are proportional.

In theorems 3, 4, and 5, these essentials to similarity are presented as follows:

Theorem 3 is the first,
Theorem 4 is the second,
Theorem 5 is in part a combination of the two;
Theorem 6 treats of relations not specified in the definition.

Of these four theorems, the first is the most important as it is almost invariably used in proving triangles similar.

The importance of similar triangles consists in the fact that in much of the subsequent work a proportion is required. In order to obtain it—

(1) two ratios must be shown to be equal to the same thing; or

(2) a line must be shown through two sides of a triangle parallel to the third side, in which case corresponding segments of the two sides form a proportion according to theorem 1; or

(3) two triangles must be proved similar.

When similarity has been proved the proportion is obtained by an application of the second specification of the definition of similar triangles.

As will be seen, a proportion is obtained by the second method in theorems 10 and 11, but in many of the theorems which follow, a proportion is obtained by the third method.

Whenever similarity is essential in demonstration, it can usually be proved by mutual equiangularity.

Indications of Homologous Parts. When similarity has been proved, every proportion of homologous sides must invariably be written so as to show that homologous sides have been used.

This can be accomplished more easily by first writing the proportion in skeleton form, as follows:

$$\frac{\text{opp}^{*}}{\text{opp}} = \frac{\text{opp}}{\text{opp}}$$

which may be filled out in the following form:

$$\frac{AB \text{ opp } 1}{AC \text{ opp } 1'} = \frac{OR \text{ opp } 3}{OT \text{ opp } 3'}$$

In the above illustration the angle notation is the same as that for equal triangles.

V THEOREM 7

The corresponding altitudes of similar triangles are in the same ratio as any two corresponding sides.

Draw the given altitudes lighter than the sides of the triangles.

Prove the similarity of two of the correspondingly situated right triangles into which the given triangles are divided by the altitudes.

V THEOREM 8

The corresponding intercepts of parallels intersected by a series of transversals passing through a common point, are proportional.

Draw two parallels and three transversals.

---

* Meaning opposite.

V THEOREM 9

A series of non-parallel transversals which cut off proportional intercepts on a series of parallels, pass through a common point if produced in the direction of their convergence.

For convenience draw three converging transversals and two parallels.

Will two of the transversals meet at a common point if produced?

Specify the construction.

Connect with the construction point, the point of intersection of the nearest parallel with the unproduced transversal; prolong the connecting line if necessary, to cut the farthest parallel.

State the relation of the intercepts of the parallels cut by the three lines which meet in the construction point.

If necessary, transform the resulting expression and the hypothesis by one of the theorems in proportion so that one member of each equation may be eliminated by applying axiom 1.

Transform the resulting equation so as to be able to prove that the third transversal coincides with the construction transversal.

V THEOREM 10

A bisector of an angle of a triangle, produced if necessary, divides the opposite side of the triangle into segments proportional to the sides respectively adjacent to them, i.e., the ratio of the segments equals the ratio of the sides respectively adjacent to the segments.

From the vertex of either unbisected angle draw a line parallel to the bisector and terminating in a side produced through the vertex of the bisected angle.

Write the relation of the ratio of the segments of the given side to the ratio of the segments of the produced side.

Compare with the conclusion and determine what must be substituted.

In order that substitution may be made, what extensional relation must be proved between the line and the line to be substituted?

But lines in this extensional relation can only be opposite angles in what relation?

Prove this relation of angles beginning with the angle opposite the line to be substituted, or with the angle opposite the line for which substitution is to be made, and finish the demonstration.

V THEOREM 11

If an exterior angle of a triangle is not double the opposite interior angle, a produced bisector of that exterior angle meets the opposite side (produced in one direction or the other) in a point so located that the ratio of its distances from the respective extremities of the produced side, equals the ratio of the sides of the given triangle respectively adjacent to those distances.

From the vertex through which the side is produced to meet the bisector, draw a line parallel to the bisector and terminating in the opposite side.

State the relation of the ratio of the segments of that side to the ratio of the segments of the produced side.

Compare the result with the conclusion and transform it by one of the theorems in proportion, so as to obtain as many of the terms required in the conclusion as possible.

Refer again to the conclusion and finish the demonstration.

V THEOREM 12

In a right triangle a perpendicular from the vertex of the right angle to the hypotenuse

(1) divides the triangle into two triangles similar to the given triangle and similar to each other;

(2) is a mean proportional to the segments of the hypotenuse.

(3) Each side of the right angle is a mean proportional to the hypotenuse and the segment of the hypotenuse adjacent to the side.

V THEOREM 13

A perpendicular to the diameter from any point in the circumference of a circle is a mean proportional to the segments into which it cuts the diameter.

V THEOREM 14

The square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides.[*]

From the vertex of the right angle draw a perpendicular to the opposite side.

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[*] This theorem is known as the theorem of the algebraic square of the hypotenuse.

Problem 3. Given a chord of an arc and the rise. Required the radius of the circle in terms of the chord and the rise.

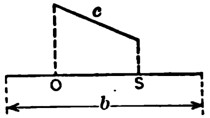

Fig. 2.

Orthogonal Projection. In Fig. 2, *OS* is the projection of *c* on *b*.

As will be seen from the figure it is the portion of *b* included between the feet of the perpendiculars to *b* from the extremities of *c*.

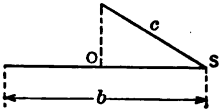

Fig. 3.

If *c* and *b* meet as in Fig. 3, the projection *OS* is the segment of *b* lying between the point in which they meet and the foot of a perpendicular from the other extremity of *c*.

In Fig. 4, *b* is projected on *c*. Its projection *OS*, is the portion of *c* produced which lies between the point in which *c* and *b* meet, and the foot of a perpendicular from the other extremity of *b*.

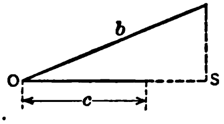

Fig. 4.

The projection of one of two intersecting lines on another is determined by perpendiculars from the extremities of the line to be projected.

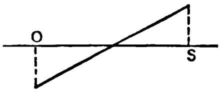

Fig. 5.

In geometry, projection is always orthogonal, that is, at right angles to the line *on which projection is made.*

The two following theorems in which projection is employed, are very important because of their use in the demonstration of the Law of Cosines* in trigonometry.

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* See Marsh's Trigonometry, page 126.

V THEOREM 15

The square of the side opposite an acute angle of any triangle equals the sum of the squares of the other two sides minus twice the product of one of the two and the projection of the other upon it.

Express as an equation the value of the projection of the side opposite the *acute* angle, in terms of the whole side upon which it is projected and the other segment of that side.

Square this equation and add to the result a quantity which will make the first member equal to the square of the side opposite the acute angle, observing theorem 14.

V THEOREM 16

The square of the side opposite the obtuse angle of a triangle equals the sum of the squares of the other two sides plus twice the product of one of the two and the projection of the other upon it.

Express algebraically the value of the projection of the side opposite the *obtuse* angle in terms of the segments of the side upon which it is projected.

V THEOREM 17

The sum of the squares of two of the sides of a triangle equals twice the square of half the third side plus twice the square of the median upon the third side.

CASE I. A triangle in which the median is drawn to the third side from the vertex formed by two *equal* sides, CASE II. A triangle in which the median is drawn to the third side from the vertex of the angle formed by two *unequal* sides.

In Case II, first consider the two angles formed by the median with the third side, proving one greater than the other, and therefore one obtuse and the other acute.

Considering the triangles into which the median divides the given triangle, apply theorems 15 and 16 to the two sides, the sum of whose squares is desired, making all projections upon the side to which the median is drawn.

V THEOREM 18

The square of the longer of two of the sides of a triangle minus the square of the shorter, equals twice the product of the third side into the projection of the median upon that side.

Demonstrate in the same manner as the preceding theorem.

V THEOREM 19

The median upon any side of a triangle equals one-half the square root of the difference between twice the sum of the squares of the other two sides and the square of the third side.

Write the sum of the squares of the sides which form the angle from whose vertex the median is drawn, expressing half the side as the whole side over 2, and solve the equation for the median.

V THEOREM 20

The product of two of the sides of any triangle equals the diameter of the circumscribed circle times the altitude upon the third side.

From the vertex of the angle included by the two given sides, draw a diameter and connect its extremities with either extremity of the third side.

Observe the number of degrees in the angle formed by the connecting line and one of the two sides.

In order to obtain the two equal products of two factors each, named in the conclusion, form a proportion by proving the similarity of the two triangles whose sides are required in the conclusion.

V THEOREM 21

If two chords intersect within a circumference the product of the segments of one equals the product of the segments of the other.

Connect the extremity of one chord with the extremity of the other.

V THEOREM 22

The sum of the squares of the sides of a parallelogram equals the sum of the squares of the diagonals.

V Theorem 23

If the bisector of an angle of any triangle terminates in the opposite side, the square of the bisector equals the product of the sides including the angle, minus the product of the segments into which the bisector cuts the third side.

Circumscribe a circle about the triangle and extend the bisector to the circumference.

Connect the point of intersection of the circumference and bisector with either extremity of the side opposite the bisected angle.

Bearing in mind the conclusion, prove a similarity of triangles and form a proportion, two of whose terms are sides of the bisected angle.

Compare this proportion with the conclusion, express the produced bisector in terms of its segments, and make the other necessary substitution by applying theorem 21.

V Theorem 24

If two secants from a point without a circle terminate in the circumference, the product of one secant and its external segment equals the product of the other secant and its external segment.

Connect the extremity of each external segment in the circumference with the extremity of the other secant in the circumference and prove the similarity of the triangles thus formed.

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V THEOREM 25

If a tangent and a secant from the same point without a circle terminate in the circumference, the tangent is a mean proportional to the secant and the external segment of the secant.

Join the point of tangency to each of the points of intersection of the secant with the circumference.

Prove the similarity of the triangles whose sides are required in the conclusion.

PROBLEM 4. *To divide a straight line in extreme and mean ratio.*

At one extremity of the line erect a perpendicular equal to half the line.

From the extremity of the perpendicular not in the line, with a radius equal to the perpendicular describe a circle.

From the other extremity of the given line draw a secant through the center of the circle, terminating in the circumference.

From the same extremity of the given line as a center, with a radius equal to the external segment of the secant describe an arc cutting the given line.

This arc divides the given line in extreme and mean ratio.

Express the ratio as an equation by referring to definition 93.

Prove by applying theorem 25, making the secant the first term of the proportion.

Bearing in mind the construction, transform the resulting expression into the algebraic expression of the extreme and mean ratio of the line by substituting for the secant in terms of the given line and applying IV theorems 5 and 7.

V THEOREM 26

If two tangents are drawn from the same point without a circle, the line joining the center of the given circle and the point from which the tangents are drawn is bisected by the center of another circle whose circumference passes through the points of tangency and the center of the given circle.

In the given circle draw radii to the points of tangency.

To each of these radii draw perpendicular bisectors.

Show that these perpendiculars meet on the line joining the center of the given circle to the point from which the tangents are drawn.

Prove that this given point is the center of the circumference named in the theorem, and that it bisects the line.

V THEOREM 27

The altitude upon any side of a triangle equals 2 *divided by the side upon which the altitude is drawn, times the square root of the product obtained by multiplying the half sum of the three sides of the triangle by the product of the differences resulting from subtracting each of the three sides from the half sum of the three sides.*

Draw two separate figures in one of which the altitude falls upon a side, in the other upon a side produced.

Mark these as separate cases.

Denote each of the three sides by small letters corresponding to the capitals at the opposite vertices.

Denote the *half* sum by s and the altitude by h.

Express the conclusion as an equation.

In either of the two given triangles write the value of the square of one of the sides forming the angle from whose vertex the altitude is drawn, observing theorem 15 or 16.

What quantity in the equation is not required in the conclusion?

Solve the equation for this quantity and square the resulting equation, writing the square of the first member, the square of the denominator of the second member, and indicating the square of the numerator of the second member.

Write the value of the first member in terms of the other two sides of the right triangle of which it is a side.

Substitute this value in the equation and solve for h^2.

Add the terms in the second member by reducing the integral term in that member to a fraction whose denominator is the same as the denominator of the fraction in that member.

The numerator of the second member is now the difference of the squares of two quantities; therefore it is resolvable into two factors, one of which is the sum of the two quantities which are squared, the other of which is the difference of the two quantities which are squared.

Write these factors.

The terms in these factors may be grouped and enclosed in parentheses so as again to show the difference of the two squares.

Therefore group, inclose, and factor.

If the work is correct the numerator of the second member has now been resolved into four factors.

For each of the different factors it will be necessary to substitute in terms of s and the different sides, starting with the equation

$$a+b+c=2s$$

Each of the other factors may be obtained from this equation by subtracting from both members of it such quantities as will make the first member successively the same as each of the other three factors.

Substitute these values in the equation which was solved for h^2 and solve the resulting equation for h.

V THEOREM 28

If the bisector of an angle of a triangle terminates in the opposite side, the bisector equals 2 divided by the sum of the sides opposite the other angles of the triangle, times the square root of the product obtained by multiplying the product of those two sides and half the sum of the three sides by the difference between the half sum and the side opposite the bisected angle.

Denote the half sum of the three sides by s and the other parts as in the preceding theorem.

Express the conclusion as an equation.

Refer to a theorem on the square of the bisector of an angle of a triangle and write the value of the square of the bisector in the given figure.

But the two segments into which the bisector cuts the side are in what ratio? (See a preceding theorem.)

Answer the question by writing the proportion.

Take this proportion by composition, making one of the segments a term of the compounded proportion; also compound the same proportion again, making the other segment a term.

Solve each of the compounded proportions for the segment it contains.

Substitute the value of each segment in the equation of the square of the bisector, performing the operations in the numerator of the resulting fraction, and indicating by parenthesis and exponent the operations in the denominator.

Add the terms in the second member.

Remove the factors common to the terms of the numerator, resolve into factors the difference of the squares thus shown in the other factor, substitute s as in the preceding theorem, and finish the demonstration.

V THEOREM 29

The sum of the squares of the sides of any quadrilateral equals the sum of the squares of the diagonals plus four times the square of the line joining the middle points of the diagonals.

Draw a very irregular quadrilateral.

Connect the middle point of one of the diagonals with each of the vertices between which the other diagonal is drawn.

Write the sum of the squares of two of the sides of each of the triangles into which the quadrilateral is divided by the diagonal whose middle point is thus connected.

Add the two equations.

Make the necessary substitution in the second member of the resulting equation by writing the value of the sum of the squares of two of the sides of the triangle formed by the two construction lines and the other diagonal.

BOOK VI

AREAS OF POLYGONS

1. ▭s = Alts
2. ▭s = Bases
3. Any 2 ▭s
4. Area ▭
5. Area ▱
6. Area △
7. Area △ Terms Sds
8. Area Trap
9. Areas △s
10. Areas △s 1 ∠ =
11. Areas ∼ △s
12. Sq Hypot

VI THEOREM 1

Rectangles of equal altitude are in the same ratio as their bases, i.e., the ratio of the rectangles equals the ratio of their respective bases.

CASE I. The bases commensurable.

Lay off a convenient common measure on each base as many times as contained.

Through the successive points of division erect perpendiculars terminating in the opposite sides.

For form of demonstration, see III theorem 24.

CASE II. **Bases Incommensurable.** (Book VII.)

VI THEOREM 2

Rectangles of equal bases are in the same ratio as their altitudes.

Consider the rectangles to stand on the side which is their altitude. (See definition base.)

VI THEOREM 3

Any two rectangles are in the same ratio as the products of their base and altitude, i.e., the ratio of the rectangles equals the base times the altitude of one, divided by the base times the altitude of the other.

Draw rectangles of unequal bases and unequal altitudes.

Construct a rectangle having a base equal to the base of one of the given rectangles and an altitude equal to the altitude of the other given rectangle.

Number the rectangles I, II, and III.

VI THEOREM 4

The area of a rectangle equals the product of its base and altitude.

Construct II, a unity of area.
Denote the base and the altitude of II by 1.
To what is the ratio of the two rectangles equal?
But the area I equals what? (See definition area.)

VI THEOREM 5

The area of any parallelogram equals the product of its base and altitude.

On the base of the parallelogram construct a rectangle of the same altitude.

VI THEOREM 6

The area of a triangle equals one-half the product of the base and altitude.

VI THEOREM 7

The area of a triangle in terms of its three sides equals the square root of the product obtained by multiplying half the sum of the three sides by the product of the differences resulting from subtracting each side separately from the half sum.

Let A equal the area, s equal the half sum, and h equal the altitude.

Denote the length of each side by a small letter corresponding to the capital at the vertex of the opposite angle.

To write the conclusion make a long radical following the equality sign.

See V theorem 27.

VI THEOREM 8

The area of a trapezoid equals one-half of the sum of the parallel sides times the altitude.

Draw a diagonal.

VI THEOREM 9

The areas of triangles are in the same ratio as the ratio of the product of the base and altitude of one, to the product of the base and altitude of the other.

VI THEOREM 10

The areas of triangles having an angle of one equal to an angle of the other, are in the same ratio as the ratio of the product of the sides including the given angle of the first triangle, to the product of the sides including the equal angle of the second triangle.

Superpose so that the equal angles are in coincidence, representing by a construction line the third side of the superposed triangle.

Connect either extremity of the construction side with the opposite vertex of the other given triangle.

Consider the given triangles as now superposed, to stand on one of the sides including the equal angles.

Draw one very light construction line as the common altitude of the two triangles whose bases are the directionally coinciding sides.

To what is the ratio of the areas of the triangles equal in terms of bases and altitude?

In like manner consider the given triangles to stand on the other sides including the equal angles.

Draw a light construction line as the common altitude of the two triangles on this base.

To what is the ratio of the two triangles equal in terms of the bases and construction altitude?

Refer to the conclusion and finish the demonstration.

VI Theorem 11

The areas of similar triangles are in the same ratio as the squares of any homologous sides.

Draw the altitudes upon any two homologous sides, and write the value of the ratio of the areas of the two given triangles, considering them to stand on the sides to which the altitudes are drawn.

But the altitudes are in what ratio?

Determine the necessary substitution by reference to the conclusion.

VI Theorem 12

The square described on the hypotenuse of a right triangle equals the sum of the squares described on the other two sides.

Use heavy lines for the sides of the given triangle and light lines for the squares.

From the vertex of the right angle draw a line to each opposite vertex of the square described on the hypotenuse.

From the same point draw a very light perpendicular (use red ink) to the side (of the square) which terminates in the opposite vertices just named.

Connect the vertex of each acute angle with the vertex of an angle of the square described on the side opposite that acute angle.

Prove the equality of the triangles in each of the two pairs of triangles having their vertices in the extremities of the hypotenuse, the sides of each of which are respectively a construction line, a side of the given triangle, and a side of a square.

Compare the areas of the triangles in each pair of triangles, with the areas of the construction rectangle and the square whose respective bases and altitudes are the same as the respective bases and altitudes of the triangles.

"The studie thereof requireth attentiveness, and such a minde as will not soone be carried to any publike shew before his full ripenes, but will rest in solitarie contemplation till he finde himselfe flidge."—MULCASTER.

"The art of moulding our fortunes consists in depending not on circumstances but on reason, and not on the reason of another but on our own."—COMENIUS.

"Another advantage of this method is that we do not accustom ourselves to a servile submission to the authority of others, but by exercising our reason every day become more ingenious in discovering the relations of things and in connecting our ideas together."—ROUSSEAU.

BOOK VII

REGULAR POLYGONS

1. Polys Same No ∼ △s
2. ∼ Polys Comp
3. Ratio Perims ∼ Polys
4. Polys ∼ 3d
5. Ratio Areas ∼ Polys
6. ⊙ Circ and Inscr Reg Poly
7. =lat Inscr Poly
8. Pt Inters Bis 2 ∠s Reg Poly
9. Cent ∠ Reg Poly $= \frac{360^\circ}{n}$
10. Cent ∠ Reg Poly Bis
11. Rad Reg Poly Bis ∠
12. ∠ Reg Poly Supp
13. Circ Div
14. Reg Polys Same No Sds
15. Rat Areas Reg Polys Same No Sds
16. Rat Perims Reg Polys Same No Sds
17. Areas Reg Polys Same No Sds Pro R^2
18. Area Reg Poly
19. Sd Reg Inscr Poly
20. Sd Reg Inscr Hex
21. $\pi = 3.14159$
22. Circ $= 2\pi R$
23. Sd Inscr =lat △ $= R\sqrt{3}$

24. Sd Inscr Square
25. Sd Reg Inscr 8-gon
26. Sd Reg Inscr = lat △
27. Apoth Inscr = lat △
28. Area Circ □
    - Prob 5. Div Circ 2, 4, 8, . . .
    - Prob 6. Div Circ 3, 6, 12, . . .
    - Prob 7. Div Circ 5, 10, 20, . . .
    - Prob 8. Div Circ 15, 30, 60, . . .
29. Sd Reg Inscr 10 − gon
30. Sd Reg Inscr 5 − gon
31. 2 Diags Reg 5 − gon
32. Area Ring
33. Lts Always = Vbls
34. Rat 2 Circums
35. Rat $\sim$ ⌒s
36. Area ⊙ $=\frac{1}{2} CR$
37. Area ⊙ $=\pi R^2$
38. Area Sect
39. Rat Areas $\sim$ Sects
40. Rat Areas $\sim$ Segs

VII THEOREM 1

Polygons composed of the same number of respectively similar triangles, correspondingly placed, are similar.

Refer to definition 79 and demonstrate.

VII THEOREM 2

Write this theorem as the converse of theorem 1.

Connect with the non-adjacent vertices, the vertices of any two angles which are equal.

VII THEOREM 3

The perimeters of similar polygons are proportional to any two corresponding sides.

Express the relation of all the corresponding sides of the polygons and obtain the ratio of the perimeters by applying one of the theorems in proportion.

VII THEOREM 4

Polygons similar to another polygon are similar to each other.

VII THEOREM 5

The areas of similar polygons are in the same ratio as the squares of any two corresponding sides.

Since the polygons are similar, of what figures in what relation are they composed? (Write answer at bottom of the page.)

Represent this by construction lines and apply VI theorem 11 to each pair of construction figures.

Through the definition of similar polygons, show the equality of the second members of the equation thus formed.

What is the relation therefore of the first members?

Apply IV theorem 12 and finish the demonstration.

VII THEOREM 6

A circle can be circumscribed about and inscribed in a regular polygon, the circles being concentric.

To prove that a circle may be circumscribed, pass a circumference through any three adjacent vertices of the polygon.

To prove that this circumference will pass through the other vertices of the polygon, draw radii and prove the equality of two triangles not having a common side, one of these triangles being one through two of whose vertices the circumference was made to pass.

VII THEOREM 7

Every equilateral, inscribed polygon, is regular.

Make use of no construction, and prove by equal arcs and inscribed angles; or, draw radii and prove by equal triangles.

VII THEOREM 8

The point of intersection of the bisectors of any two of the angles of a regular polygon, is equally distant from the vertices of the polygon.

Connect the point of intersection with the vertices.

VII THEOREM 9

An angle at the center of a regular polygon equals 360° *divided by the number of sides of the polygon.*

In the hypothesis and conclusion denote the number of sides by *n*.

Connect the center with the other vertices.

VII THEOREM 10

An angle at the center of a regular polygon is bisected by the apothem.

VII THEOREM 11

A radius of a regular polygon bisects the angle at the vertex.

VII THEOREM 12

An angle of a regular polygon is the supplement of an angle at the center of the polygon.

Write the value of an angle at the center in terms of a straight angle.

In the same terms write the value of the given angle of the polygon.

VII THEOREM 13

If a circumference is divided into any number of equal arcs,

(1) their chords form a regular inscribed polygon;

(2) the tangents at the points of division form a regular circumscribed polygon of the same number of sides as the inscribed polygon.

Interpret both conclusions in the language of the definitions which may apply, and demonstrate by proving the relations thus specified.

A simple demonstration for (2) is one based on III theorem 28.

VII THEOREM 14

Regular polygons having the same number of sides are similar.

Interpret the conclusion by definition 79.

VII THEOREM 15

The ratio of the areas of regular polygons of the same number of sides equals the ratio of the squares of any two corresponding sides.

What is the relation of the polygons to each other? In what relation therefore, are the squares of their homologous sides?

VII Theorem 16

The ratio of the perimeters of regular polygons of the sam number of sides equals the ratio of their radii; also the rati of their apothems.

Denote the number of sides by *n*.

VII Theorem 17

The ratio of the areas of regular polygons of the same number of sides equals the ratio of the squares of the radii of the circumscribed circles; also the ratio of the squares of the radii of the inscribed circles.

VII Theorem 18

The area of a regular polygon equals one-half the product of the perimeter and the apothem.

Draw radii to each of the vertices.

VII Theorem 19

The length of a side of a regular inscribed polygon having twice as many sides as another regular polygon inscribed in the same circle, equals the square root of the product obtained by multiplying by the radius, the difference between twice the radius and the square root of the difference between four times the radius squared and the square of a side of the polygon having the smaller number of sides.

Inscribe in a circle a side TV of a regular polygon.

Within the segment included by this side, inscribe a side TS of a regular polygon of double the number of sides, making the sides thus drawn coincident at T.

From T draw a radius R.

Express the conclusion as an equation. In order to do this, write separately the algebraic equivalent of each thing named in the theorem.

Draw a diameter perpendicular to TV.

Prove that one extremity of this diameter is coincident with S.

Join T with B the extremity of the diameter.

Prove that TBS is a right triangle.

Therefore what is the relation of TS to the diameter and the segment of the diameter adjacent to TS?

Substitute for the diameter in terms of the radius, and solve for the square of TS.

Compare the final equation with the algebraic expression of the conclusion.

It will be seen that it is necessary to find values for the segment of the diameter adjacent to the shorter chord, in terms of R and TV. This necessity determines all further operations.

Express the segment of the diameter adjacent to TS, in terms of the radius and the segment of the diameter between TV and the center of the circle.

By the square of the hypotenuse express the segment of the radius included between the center of the circle and TS, in terms of R and TV.

Study the conclusion to determine what transformations and substitutions are required to finish the demonstration.

VII THEOREM 20

A side of a regular inscribed hexagon is equal to the radius of the circle.

Connect with the center any two adjacent vertices.

Ratio of the Circumference to the Diameter. As proved by the theorem regarding the ratio of circumferences, the ratio of the circumference of a circle to the diameter is the same in all circles. This ratio is denoted by π which is the Greek p, pronounced as though spelled pi with the i long.

The attempts of the ancients and some comparatively modern mathematicians to find an exact value for π are of great historical interest. Even after it had been proved that the circumference and the diameter are incommensurable, the attempts were continued and the value of π has consequently been determined to 707 decimal places.

In the next theorem it will be proved that the 5-place value of π is 3.14159.

VII THEOREM 21

The approximate ratio of the circumference of a circle to the diameter equals 3.14159.

Draw a unit circle, denote the ratio by π, and express the conclusion algebraically.

In the unit circle inscribe one side each of a regular hexagon and dodecagon, coincident at one extremity.

By VII theorem 19 express the side of the dodecagon S_{12} in terms of R and the side of the hexagon S_6.

Substitute for the side of the hexagon in terms of the radius and solve the resulting equation for the numerical value of the side of the dodecagon to eight decimal places, *as an indicated root.* (Do not extract the root in the result.)

Substitute in the same formula the numerical value of the side of the dodecagon thus formed, and compute to eight decimal places *as an indicated root,* the numerical value of the side of a regular inscribed polygon of twenty-four sides.

In like manner successively substitute until the length of a side of a regular inscribed polygon of 384 sides has been found.

Finish the demonstration.

VII THEOREM 22

A circumference equals $2\pi R$.

VII THEOREM 23

A side of an inscribed equilateral triangle equals the radius times the square root of 3.

Denote the radius by R.

From any vertex draw a diameter and connect its extremity with one extremity of the side which it intersects.

Prove the connecting line equal to R.

VII THEOREM 24

A side of an inscribed square is equal to the radius times the square root of 2.

Draw a diagonal.

VII THEOREM 25

A side of a regular inscribed octagon is equal to the radius times the square root of the difference between 2 and the square root of 2.

Apply VII Theorem 19.

VII THEOREM 26

A side of an inscribed equilateral triangle equals one-half the side of a circumscribed equilateral triangle.

VII THEOREM 27

The apothem of an inscribed equilateral triangle equals one-half the radius.

VII THEOREM 28

The area of a circumscribed square is double the area of an inscribed square.

Draw the figure so that the sides of the circumscribed square are tangent at the vertices of the inscribed square.

Division of the Circumference. The divisions of the circumference in the following problems are given in Euclid, and were the only ones known in elementary geometry until the publication in 1801, of the celebrated arithmetic of Gauss. In this work is proved the possibility of dividing the circumference by methods of elementary geometry, into 17, 257, and in general into any number of equal parts expressed by 2^n+1 provided *n is integral* and 2^n+1 is a *prime number, n* denoting the number of sides.

For example, 2^0+1, 2^1+1, 2^2+1, each being a prime number, give the divisions of the circumference in problems 5, 6, and 7 respectively. 2^3+1 is not a prime number and it is not possible therefore by methods of elementary geometry to divide a circumference into nine equal arcs.

The method of dividing the circumference into seventeen equal arcs represented by 2^4+1, is too difficult for elementary geometry. It is treated at length in Klein's "Famous Problems of Elementary Geometry" and has historical value only.

In the analytieal geometry of Marsh's "Constructive Text-Book" is a discussion of the chordal invented by Mr. J. Bruen Miller in 1880. By means of this curve the circumference may easily be divided into any required number of equal arcs. The construction and use of this curve are so simple that it may well supercede other mathematical methods of dividing the circumference.

A practical method is to use a protractor, but by the geometric method which is required in the following problems, only the compasses and the straight edge are permitted.

PROBLEM 5. *To divide a circumference into 2, 4, 8, 16, etc., equal arcs.*

Construct and prove.

PROBLEM 6. *To divide a circumference into 3, 6, 12, 24, etc., equal arcs.*

Construct and prove.

PROBLEM 7. *To divide a circumference into 5, 10, 20, etc., equal arcs.*

Draw a radius of the circle and divide it in extreme and mean ratio, laying off the greater segment from the center, From the other extremity of the radius, draw a chord equal to the greater segment.

The arc subtended by the chord is $\frac{1}{10}$ of the circumference.

Prove as follows: Connect the extremity of the chord with the center.

Connect the extremity of the chord also with the point at which the radius is divided into greater and lesser segments.

Consider the two triangles, one of which is formed by the two radii and the chord, the other by the chord, lesser segment of the radius and line CR joining the point of division of the radius with the extremity of the chord.

Prove that these two triangles have an angle of one equal to an angle of the other and the including sides proportional, and are therefore similar.

The necessary proportion can be obtained by substituting the chord in the proportion expressing the extreme and mean ratio in which the radius was divided.

The triangle, two of whose sides are the radii, is what kind of a triangle?

What angles are therefore equal?

Therefore the smaller triangle which is similar to it has what angles equal? (Prove by the indirect method.)

Prove CR equal to the radius.

Finish the demonstration by proving that the sum of the angles of the smaller triangle equals five times the angle between the two radii.

PROBLEM 8. *To divide a circumference into 15, 30, 60, etc., equal arcs.*

From the arc subtended by the side of a regular hexagon, subtract geometrically the arc subtended by the side of a regular decagon.

VII THEOREM 29

A side of a regular inscribed decagon equals one-half the radius times the difference between the square root of 5 and 1.

Use the same figure as in VII Problem 7.

In the proportion expressing the extreme and mean ratio in which the radius was divided, substitute the side of the decagon for the greater segment of the radius.

In the same proportion substitute for the shorter segment in terms of R and the side of the decagon.

VII THEOREM 30

A side of a regular inscribed pentagon equals one-half the radius times the square root of the difference between 10, and 2 times the square root of 5.

VII THEOREM 31

Two diagonals of a regular pentagon, not drawn from the same vertex, divide each other in extreme and mean ratio.

Prove the similarity of the two triangles one side of one of which is a diagonal, and two of the sides of the other are the shorter segments of the diagonals.

VII THEOREM 32

The area of the ring included between the circumferences of two concentric circles, equals the area of a circle whose radius equals one-half the chord of the outer circle drawn tangent to the inner circle.

Draw a radius perpendicular to the chord.

Denote by R the radius of the inner circle, and by $R+z$ the radius of the outer circle.

Formulate the area of the ring in terms of the radii of the concentric circles.

By expansion, cancellation, and factoring, reduce the second member of the formula to its simplest form.

Produce $R+z$ to the circumference of the outer circle and join the extremities of the resulting diameter, with one extremity of the chord.

Prove one-half the given chord $=\sqrt{z(2R+z)}$

The Theory of Limits and its Application to the Determination of the Ratio of Incommensurables

A New Classification of Quantities. The child beginning number work, for some time knows only integers. When a degree of skill in their use has been acquired, he is told of another kind of numbers, called decimals and fractions. In elementary algebra the classification is into known and unknown and into rational and surd.

At this point in the study of mathematics a new and radically different classification is a necessity. Not only this, but a new idea of number must be entertained. Our conception up to this time has been of numbers which differ by finite, expressible increments, as 1, 2, 3, or 1.2, 1.3, or $\frac{1}{3}$, $\frac{1}{2}$, $\frac{3}{4}$, etc. Such numbers are called discontinuous because one number does not pass into another except by a sudden, measurable addition, positive or negative.

Even when this is made, we do not think of 2 having become 1, or of 3 having become 5, but rather that we now have 5 instead of 3 and that this has been effected by definite, external increase.

Opposed to this artificial conception is the natural one of number as continuous. The thermometer which an hour ago was at 70° and now at 75°, the vine which has grown ten inches within a week, the tree, the man, attained their present state not discontinuously, but by imperceptible, indefinitely minute, immeasurable, internal increase.

The processes of nature are more or less continuous, and only time renders possible their expression in a discontinuous number system.

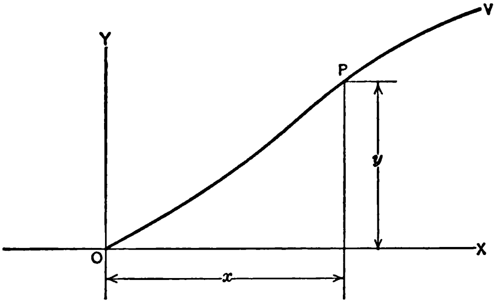

FIG. 6.

A Variable. In the figure, **OX** and **OY** are two straight lines at right angles, called axes.

OV represents the path traced by the continuously moving point *P*, in passing from *O* to *V*.

The distance of the point from *OY* at any moment is denoted by x, and its distance from *OX* by y.

As the point moves continuously, x and y change in value continually.

A **variable** is a quantity whose value changes continually.

x and y are therefore variables whose values at any moment are determined by the position of the moving point.

Another Illustration of a Variable.

$$x=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\tfrac{1}{64}+\cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

If the above series is continued uninterruptedly by additions of $\frac{1}{2}$ each preceding fraction, the value of x changes continually. x is therefore a variable.

A Constant. In the preceding paragraph, x not only changes its value but continually approaches a fixed, constant value of 2 and may be made as nearly equal to 2 as is desired.

In other words, 2 is the value which the variable x continually approaches. 2 does not change in value, but x changes.

A **constant** is a quantity whose value does not change.

A Limit. In geometry a limit is:

(1) a constant,

(2) the constant which a variable continually approaches,

(3) the constant whose difference from a variable approaches zero indefinitely.

Therefore when the difference between a constant and a variable may approach zero indefinitely, the constant is the *limit* of the variable.

In the series

$$x=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\tfrac{1}{64}+\cdot\ \cdot\ \cdot\ \cdot$$

2 is therefore the limit of x.

> "Thou shalt not knowledge gain
> Of this glad being, if thine eyes are held
> Still in the depths below."—DANTE.

VII THEOREM 33

THEORY OF LIMITS. *The limits of always equal variables are equal.*

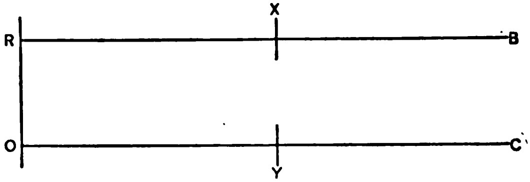

FIG. 7.

Given Vbl RX always $=$ Vbl OY

$RX \doteq {}^{*}RB$

$OY \doteq OC$

Prove $RB = OC$

Demonstrate as follows:

Assume RB *greater* than OC.

Mark off on RB a distance RH equal to OC.

But RX is what and RB is what?

Therefore how near may RX approach RB?

Suppose RX to have become RF, greater than RH.

But RX always equals what?

Therefore OY has become greater than what?

But RH equals what? Substitute.

But can OY be greater than OC? (See definition limit.)

---

* $\doteq$ denotes "approaches as a limit."

(1) Therefore is *RB* greater than *OC*?

Suppose *RB* *less* than *OC*.
Then on *OC* lay off *OW* equal to *RB*.
But *OY* is what and *OC* is what?
Therefore how near may *OY* approach *OC*?
Suppose *OY* to have become *OS*, greater than *OW*.
But *OY* always equals what?
Therefore *RX* has become greater than what?
But *OW* equals what? Substitute.
But can *RX* become greater than *RB*?
(See definition limit.)
(2) Therefore is *RB* less than *OC*?

Therefore *RB* is in what relation to *OC*?
(As authority, give (1), (2), and Magnitude Axiom.)

Incommensurable Quantities. These are quantities which have no common measure; therefore their ratio cannot be determined by division into parts by a common unit but is expressed only by the application of the theory of limits.

Common examples of incommensurable quantities are the side of a square and its diagonal, the circumference of a circle and its diameter, an arc in degrees and in radians, a meter and a yard.

Some Deferred Demonstrations. The demonstrations immediately following are for the incommensurable cases of III theorem 24, V theorem 1, and VI theorem 1.

The method and order of demonstration required by the following suggestions for demonstrating III theorem 24, Case II, should be observed in every detail and used as a model in all applications of the theory of limits.

III THEOREM 24. *Angles at the center of equal circles or of the same circle are in the same ratio as their intercepted arcs, i.e., the ratio of the angles equals the ratio of their respective arcs.*

CASE II: $\frown$s **AR and VS incommensurable.**

Draw the figure as in Case I, except that AR and VS are *incommensurable.*

Demonstrate as follows:

AR and VS are what kind of quantities?

Therefore they do not have what kind of a measure?

Therefore beginning at R apply to AR as many times as contained, any measure which will divide VS into equal parts.

Through X the last point of division of AR, draw a line to the center.

Express the relation of $\frac{\angle ROX}{\angle VO'S}$ to $\frac{RX}{VS}$.

Suppose the number of parts into which VS is divided to be continually increased and one of the successively resulting parts to be continually applied to AR from R.

Let XO represent the line which continually joins with the center the successively resulting last point of division of AR.

Then $\frac{\angle ROX}{\angle VO'S}$ becomes what?

(As authority give Cons and Def Var.)

And $\frac{RX}{VS}$ becomes what?

(As authority give Cons and Def Var.)

But $\frac{ROX}{VO'S}$ is always in what relation to $\frac{RX}{VS}$?

(As authority give Cons and Case I.)

Therefore limit $\frac{ROX}{VO'S}$ is in what relation to limit $\frac{RX}{VS}$?

But limit $\frac{ROX}{VO'S}$ = what?

(As authority give Cons and Def limit.)

And limit $\frac{RX}{VS}$ = what?

Finish the demonstration.

V **Theorem** 1. *If two of the sides of a triangle are cut by a line parallel to the third side, the ratio of the segments of one side equals the ratio of the corresponding segments of the other side.*

Case II: **AR and RC incommensurable.**

Draw the figure as in Case I, except that AR and RC are *incommensurable.*

AR and RC are what kind of quantities?

Therefore they do not have what kind of a measure?

Therefore beginning at R apply to AR as many times as contained, a measure which will divide RC into equal parts.

Through X the last point of division of AR, draw a line parallel to AB intersecting CB in Y.

Express as an equation the relation of $\frac{CR}{RX}$ to $\frac{CS}{SY}$.

Suppose the number of parts into which RC is divided to be continually increased and one of the successively resulting parts to be continually applied to AR from R.

Let XY represent the parallel which is continually drawn through the successively resulting last points of division of RA and SB.

Then $\frac{CR}{RX}$ becomes what?

(As authority give Cons and Def Var.)

And $\frac{CS}{SY}$ becomes what?

(As authority give Cons and Def Var.)

But $\frac{CR}{RX}$ is always in what relation to $\frac{CS}{SY}$?

(As authority give Cons and Case I.)

Therefore limit $\frac{CR}{RX}$ is in what relation to limit $\frac{CS}{SY}$?

But limit $\frac{CR}{RX}$ = what?

(As authority give Cons and Def limit.)

And limit $\frac{CS}{SY}$ = what?

Finish the demonstration.

VI THEOREM 1. *Rectangles of equal altitude are in the same ratio as their bases, i.e., the ratio of the rectangles equals the ratio of their respective bases.*

CASE II. **Bases incommensurable.**

Denote the bases by b and b', the altitude by h, and demonstrate.

"No man is master of a science by another's understanding."—XENOPHON.

VII THEOREM 34

The ratio of two circumferences equals the ratio of their radii; also the ratio of their diameters.

Inscribe a regular polygon of the same number of sides, in each circle.

Write the algebraic statement of the ratio of the perimeters of the polygons.

Finish the demonstration by applying the Theory of Limits, basing its application upon a continual doubling of the number of sides of the polygons.

VII THEOREM 35

Similar arcs are in the same ratio as their respective radii.

Draw two unequal circles having an arc of one similar to an arc of the other, denoting by a letter the number of degrees in each, as part of the hypothesis.

Express algebraically the ratio of the two circumferences to their radii.

VII THEOREM 36

The area of a circle equals one-half the circumference times the radius.

Circumscribe a regular polygon, formulate its area, and prove by the Theory of Limits, assuming the number of sides of the polygon to be doubled continually.

VII THEOREM 37

The area of a circle equals πR^2.

VII THEOREM 38

The area of the sector of a circle equals one-half the arc of the sector times the radius.

In the hypothesis name the sector as having $n°$.

To what is the area of the circle (a sector whose arc is 360°) equal?

But the given sector equals what part of the circle?

Therefore to what is its area equal in terms of the area of the circle?

Transform this equation if necessary into the equation of the conclusion.

VII THEOREM 39

The areas of similar sectors are in the same ratio as the squares of their respective radii.

Draw unequal circles.

VII THEOREM 40

The areas of similar segments are in the same ratio as the squares of their respective radii.

Draw unequal circles having a segment of one similar to a segment of the other.

Observe that each segment equals the sector minus a triangle.

In what relation are the sectors having the same arcs as the given segments?

To what therefore, is the ratio of the areas of these sectors equal?

Prove that the areas of the triangles formed by the radii and the chords of the arcs, are in the same ratio.

Formulate the relation of the ratio of the areas of the sectors to the ratio of the areas of the triangles.

SOLID GEOMETRY

Book VIII. Lines and Planes
Dihedral Angles
Polyhedral Angles

Book IX. Polyhedrons
  The Prism
  The Pyramid
The Cylinder
The Cone

Book X. The Sphere

Axioms

1 to 25. The general and the geometric axioms as given on pages 9–11, 37–38, 40, and 43, in plane geometry.

26. **Thru St Inf No Pls.*** Through a straight line an infinite number of planes may be passed.

27. **Pl Ax.** Only one plane can be passed through:
    (1) 1 Pl Thru 3 Pts
        Three points not in the same straight line;
    (2) 1 Pl Thru Line and Pt
        A line and a point without the line;
    (3) 1 Pl Thru 2 Inters Lines
        Two intersecting lines;
    (4) 1 Pl Thru 2 || Lines
        Two parallel lines.

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* The abbreviated form preceding each axiom is suggested for use when the axioms are cited as authority in demonstration.

DEFINITIONS

SOLID GEOMETRY

121. **A Geometric Solid** is a limited portion of unoccupied space of definite form, whose parts are not all in the same plane.

122. **The Intersection of Two Planes** is the line containing all the points common to the planes, but no other points.

123. **A Line Perpendicular to a Plane** is a line perpendicular to all lines in the plane, through its point of intersection with the plane.

124. **A Line Oblique to a Plane** is a line perpendicular to only one line in the plane, through its point of intersection with the plane.

125. **The Foot of a Perpendicular to a Plane** is its point of intersection with the plane.

126. **A Line Parallel to a Plane** is a line parallel to its projection on the plane. It therefore will not meet the plane even when produced.

127. **Parallel Planes** are planes which do not meet within a finite distance, even when produced.

128. **A Solid Figure** is a figure whose parts are in more than one plane.

129. **A Solid Angle** is a solid figure formed by two or more planes which meet.

130. **A Dihedral Angle** is a solid angle formed by two planes.

131. **The Edge of a Dihedral Angle** is the line in which the planes meet.

132. **The Faces of a Dihedral Angle** are the planes forming the angle.

133. **The Plane Angle of a Dihedral Angle** is the angle formed by two lines, one in each of the planes forming the dihedral angle, each perpendicular to the edge at the same point.

134. **Equal Dihedral Angles** are those which coincide or can be made to coincide.

135. **Adjacent Dihedral Angles** are those having a common face and a common edge.

136. **A Right Dihedral** is a dihedral angle whose plane angle is a right angle.

137. **A Plane, Perpendicular to Another Plane,** is a plane which forms a right dihedral angle with that plane.

138. **The Projection of a Point on a Plane** is the foot of a perpendicular from the point to the plane.

139. **The Projection of a Line on a Plane** is that part of a line in the plane, included between the feet of the perpendiculars to the plane from each extremity of the given line.

140. **The Angle of Inclination of a Line to a Plane** is the angle formed by the line with its projection on the plane.

141. **A Polyhedral Angle** is a solid angle formed by three or more planes which meet in a common point.

142. **The Vertex of a Polyhedral Angle** is the point in which the planes meet.

143. **The Edges of a Polyhedral Angle** are the lines in which the planes forming the polyhedral angle intersect.

144. **The Faces of a Polyhedral Angle** are the portions of the planes included between its edges.

145. **A Face Angle of a Polyhedral Angle** is an angle formed by two intersecting edges.

146. **The Parts of a Polyhedral Angle** are its face angles and its dihedral angles.

147. **A Convex Polyhedral Angle** is a polyhedral angle whose every section made by a plane cutting all the edges, is a convex polygon.

148. **A Convex Polygon** is a polygon no side of which, although produced, will enter the polygon.

149. **A Trihedral Angle** is a polyhedral angle having three faces.

150. **A Polyhedron** is a solid figure bounded by planes.

151. **The Faces of a Polyhedron** are the planes which bound it, as limited by each other.

152. **The Edges of a Polyhedron** are the lines in which the faces intersect.

153. **The Basal Edges of a Polyhedron** are the edges in which the base intersects the faces.

154. **The Vertices of a Polyhedron** are the points in which the edges intersect.

155. **A Diagonal of a Polyhedron** is a line joining two vertices in different faces.

156. **A Tetrahedron** is a polyhedron having four equal faces.

157. **A Hexahedron** is a polyhedron having six equal faces.

158. **An Octahedron** is a polyhedron having eight equal faces.

159. **A Dodecahedron** is a polyhedron having twelve equal faces,

160. **An Icosahedron** is a polyhedron having twenty equal faces.

161. **A Convex Polyhedron** is a polyhedron whose every section made by a plane, is a convex polygon.

162. **A Prism** is a polyhedron whose lateral faces are parallelograms, and whose other two faces, called bases, are parallel to each other.

163. **A Section of a Prism** is a polygon formed by the intersection of a plane with the faces of the prism.

164. **A Right Section of a Prism** is a section formed by a plane perpendicular to the lateral edges.

165. **The Lateral Area of a Prism** is the area of its lateral faces.

166. **Equal Prisms** are those which coincide or which can be made to coincide.

167. **An Oblique Prism** is a prism whose lateral edges are oblique to its bases.

168. **A Right Prism** is a prism whose lateral edges are perpendicular to its bases.

169. **The Altitude of a Prism** is the perpendicular distance between the bases.

170. **A Truncated Prism** is either portion into which the prism is divided by a plane not parallel to its bases.

171. **A Triangular Prism** is a prism whose base is a triangle.

172. **Equivalent Solids** are

(1) solids of equal volume but of different shape:

(2) solids having equal bases and altitudes respectively, but with parts arranged in a different order.

173. A **Parallelepiped** is a prism whose base is a parallelogram.

174. A **Right Parallelepiped** is a parallelepiped whose lateral edges are perpendicular to its bases.

175. A **Rectangular Parallelepiped** is a right parallelepiped whose bases are rectangles.

176. **The Dimensions of a Rectangular Parallelepiped** are three of its edges which meet at any vertex.

177. **The Volume of a Rectangular Parallelepiped** is its ratio to a unit rectangular parallelepiped (a unit of volume), each of whose dimensions is one linear unit.

178. A **Cube** is a rectangular parallelepiped whose dimensions are equal.

179. A **Pyramid** is a polyhedron whose base is a polygon and whose lateral faces are triangles having a common vertex called the apex of the pyramid.

180. **The Lateral Edges of a Pyramid** are the lines of intersection of the lateral faces.

181. **The Lateral Area of a Pyramid** is the area of the lateral faces.

182. **The Altitude of a Pyramid** is the distance from the apex to the base.

183. A **Regular Pyramid** is a pyramid whose base is a regular polygon in whose center will lie the projection of the apex of the pyramid.

184. **The Slant Height of a Regular Pyramid** is the altitude of a lateral face.

185. A **Frustum of a Pyramid** is the portion of the pyramid remaining after the top is cut off by a plane parallel to the base.

186. **A Cylindrical Surface** is a curved surface generated by a line called the generatrix, moving parallel to itself in continuous contact with, but not in the plane of, a curve called the directrix.

187. **An Element of the Surface** is a line representing the generatrix in any position.

188. **A Cylinder** is a geometric solid whose lateral surface is cylindrical and whose two basal surfaces are plane and parallel.

189. **The Altitude of a Cylinder** is the perpendicular distance between its bases.

190. **A Right Cylinder** is a cylinder whose elements are perpendicular to its bases.

191. **An Oblique Cylinder** is a cylinder whose elements are oblique to its bases.

192. **A Circular Cylinder** is a cylinder whose bases are circles.

Elementary geometry treats of circular cylinders only. By cylinder, therefore, as used in geometry, a circular cylinder is understood.

193. **A Cylinder of Revolution** is a right, circular cylinder.

It may therefore be generated by the revolution of a rectangle about one side as an axis.

194. **Similar Cylinders of Revolution** are cylinders generated by similar rectangles revolving about homologous sides.

195. **A Right Section of a Cylinder** is a section made by a plane, perpendicular to its elements.

196. **A Plane, Tangent to a Cylinder,** is a plane containing only one element of the surface.

197. **The Element of Contact** is the element of the surface contained by a tangent plane.

198. **A Line, Tangent to a Cylinder,** is any line in a tangent plane, cutting the element of contact.

199. **A Prism Inscribed in a Cylinder** is a prism whose bases are inscribed in the bases of the cylinder.

200. **A Prism Circumscribed about a Cylinder** is a prism whose bases are circumscribed about the bases of the cylinder.

201. **A Conical Surface** is a surface generated by the complete revolution of a line (called the generatrix), which passes through a fixed point (called the vertex), and constantly touches a fixed curve (called the directrix) not in the same plane with the generatrix.

202. **An Element of a Conical Surface** is a line representing the generatrix in any position.

203. **The Nappes of a Conical Surface** are the portions of the conical surface on each side of the vertex.

204. **A Cone** is a geometric solid whose lateral surface is conical and whose basal face is a plane cutting all the elements.

205. **The Altitude of a Cone** is the perpendicular from the vertex to the base.

206. **A Circular Cone** is a cone whose base is a circle.

207. **The Axis of a Cone** is a line from the vertex to the center of the base.

208. **A Right Circular Cone** is a circular cone whose axis is perpendicular to its base.

209. **The Slant Height of a Cone** is a line from the vertex, perpendicular to a tangent to the base at any point.

210. **A Cone of Revolution** is a cone generated by the revolution of a right triangle about one of the sides including the right angle.

211. **Similar Cones of Revolution** are cones generated by the revolution of similar right triangles about the homologous sides including the right angles.

212. **A Plane, Tangent to a Cone,** is a plane containing only one element of the surface.

213. **A Line, Tangent to a Cone,** is a line having only one point in the surface of the cone, even when produced.

214. **A Pyramid Inscribed in a Cone** is a pyramid whose base is inscribed in the base of the cone and whose apex coincides with the vertex of the cone.

215. **A Pyramid Circumscribed about a Cone** is a pyramid whose base is circumscribed about the base of the cone and whose apex coincides with the vertex of the cone.

216. **A Frustum of a Cone** is the portion of the cone included between the base and the section formed by a plane cutting all its elements, parallel to the base.

217. **The Lower Base of the Frustum of a Cone** is the base of the cone.

218. **The Upper Base of the Frustum of a Cone** is the section formed by the parallel plane.

219. **The Altitude of a Frustum of a Cone** is the perpendicular distance between its bases.

220. **The Lateral Surface of the Frustum of a Cone** is the curved surface.

221. **The Slant Height of the Frustum of a Cone of Revolution** is the element of contact of a plane, tangent to the frustum.

222. **A Sphere** is a geometric solid bounded by a surface, all of whose points are equally distant from the center.

223. **A Sphere of Revolution** is a sphere generated by one complete revolution of a semicircle about its diameter.

224. **A Radius of a Sphere** is a line from the center to the surface of the sphere.

225. **A Diameter of a Sphere** is a line through the center, whose extremities are in the surface.

226. **A Line, Tangent to a Sphere,** is a line having all points except one, the point of tangency, without the surface even when produced.

227. **A Plane, Tangent to a Sphere,** is a plane, all of whose points except one, the point of tangency, are without the sphere even when the plane is produced.

228. **Tangent Spheres** are spheres whose surfaces have only one point in common.

229. **A Sphere in a Polyhedron** is a sphere to which each face of the polyhedron is tangent.

230. **A Sphere Circumscribed about a Polyhedron** is a sphere in whose surface lie the vertices of the polyhedron.

231. **The Polar Distance of a Circle** is the distance of its nearest pole from any point on its circumference.

232. **A Great Circle of a Sphere** is a section made by a plane which passes through the center of the sphere.

233. **A Small Circle of a Sphere** is a section made by a plane which does not pass through the center of the sphere.

234. **The Axis of a Circle of a Sphere** is the diameter of the sphere which is perpendicular to the plane of the circle. The **poles** are the extremities of the axis.

235. **Parallel Circles** are circles whose planes are parallel.

236. **The Angle of Two Curves Through the Same Point** is the angle formed by two straight lines tangent to the respective curves at the given point.

237. **A Spherical Angle** is a solid figure formed by the intersection of the arcs of two great circles of a sphere.

238. **The Plane Angle of a Spherical Angle** is the angle formed by tangents to the sides of the spherical angle at its vertex.

239. **The Number of Degrees in a Spherical Angle** is the number of degrees in its plane angle.

240. **A Spherical Polygon** is the portion of the surface of a sphere, bounded by three or more arcs of great circles.

241. **The Sides of a Spherical Polygon** are the arcs which bound it.

242. **The Angles of a Spherical Polygon** are the angles formed by the intersecting arcs.

243. **The Vertices of a Spherical Polygon** are the vertices of its angles.

244. **A Convex Spherical Polygon** is a spherical polygon whose corresponding polyhedral angle is convex.

In geometry, only convex spherical polygons are considered.

245. **The Corresponding Polyhedral Angle of a Spherical Polygon** is the polyhedral angle formed at the center of the sphere by the great circles by whose arcs the spherical polygon is bounded.

246. **A Diagonal of a Spherical Polygon** is an arc of a great circle connecting any two non-adjacent vertices.

247. **Equal Spherical Polygons** are spherical polygons which coincide or can be made to coincide.

248. **A Spherical Triangle** is a spherical polygon of three sides; like a plane triangle, it may be right, oblique, equilateral, isosceles, or scalene.

249. **A Bi-rectangular Triangle** is a spherical triangle having two right angles.

250. **A Tri-rectangular Triangle** is a spherical triangle having three right angles.

251. **A Spherical Degree** is a bi-rectangular triangle which equals one-ninetieth of a tri-rectangular triangle.

252. **The Spherical Excess of a Spherical Triangle** is the difference between the sum of its angles and 180 spherical degrees.

253. **Polar Triangles** are spherical triangles in which the vertices of either are the respective poles of the sides of the other.

254. **A Zone** is a portion of the surface of a sphere included between two parallel planes which cut the sphere.

255. **The Bases of a Zone** are the circumferences of the sections made by the planes.

256. **The Altitude of a Zone** is the distance between the planes.

257. **A Zone of One Base** is a zone having one of its bounding planes tangent to the sphere.

258. **A Spherical Pyramid** is a portion of a sphere bounded by a spherical polygon and the planes of its sides.

259. **The Vertex of a Spherical Pyramid** is the point in which its faces meet at the center of the sphere.

260. **The Base of a Spherical Pyramid** is a spherical polygon formed by the intersections of its sides with the surface of the sphere.

261. **A Spherical Sector** is a portion of a sphere, generated by the revolution of a circular sector about a radius which bisects the arc of the sector, or about any diameter.

262. **The Base of a Spherical Sector** is the zone generated by the arc of the circular sector.

263. **A Spherical Segment** is a portion of a sphere contained between two parallel planes.

264. **The Bases of a Spherical Segment** are the sections made by the parallel planes.

265. **The Altitude of a Spherical Segment** is the distance between its bases.

266. **A Spherical Segment of One Base** is a segment, one of whose planes is tangent to the sphere.

Unless otherwise specified, "spherical segment" signifies a segment of one base.

"He who
cannot see, must reach
As best he may the truth of men by help of words
They please to speak, must fare at will of who
affords
The banquet,"—so I thought. "Who
sees not, hears and so
Gets to believe."

ROBERT BROWNING.

INTRODUCTION

A Geometric Solid. Imagine any physical solid, say a pyramid, removed, and the outline preserved of the space which it formerly occupied.

The space before filled by the pyramid and now bounded by its preserved outline is a geometric solid.

Imagine the space which a pyramid would occupy.

That space which it would occupy is a geometric solid.

Imagine a pyramid of space.

This imaginary pyramid of space is also a geometric solid.

These three illustrations show that the **solids of geometry** are not material, but **ideal.**

In the first illustration the conception was largely material in that there was first imaged a material pyramid, then the pyramid removed, and finally the space it previously occupied.

In the second illustration the conception of the space which a pyramid would occupy if existent, approximates the ideal.

In the third illustration the conception of a pyramid of space with no reference to the material pyramid by which it might be occupied, is as fully as possible ideal.

Representation of a Geometric Solid. Following are figures representing planes in various relations. The principle governing their construction is that the amount of shading on any line is determined by the line's position in the figure.

Lines representing rear edges are drawn light, receiving no shading whatever. Lines representing front edges or any edge which can be seen, are drawn heavy. Lines midway between front and rear edges have medium shading.

In general, the nearer the line to the front of a figure the heavier the shading; the nearer to the rear the lighter the shading. A line hidden by a plane is shown as a dash, or dotted, line.

These principles should be observed in drawing all figures in solid geometry.

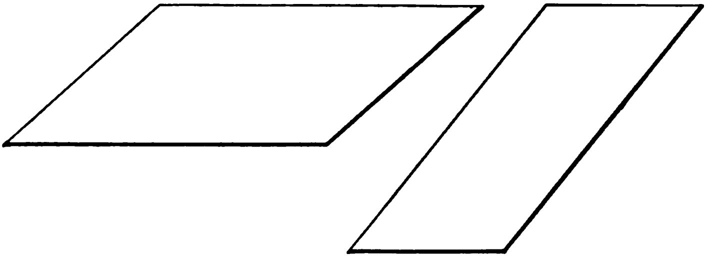


FIG. 8. Planes showing front and *right* edges.

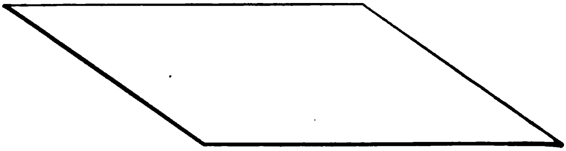


FIG. 9. Plane showing front and *left* edges.

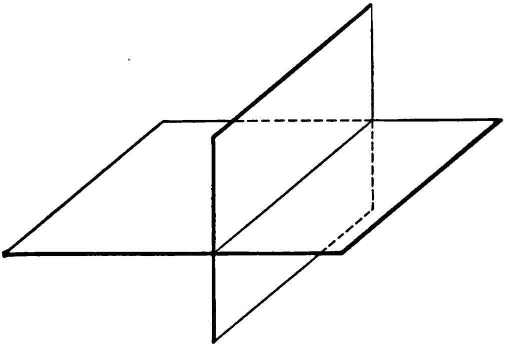


FIG. 10. Perpendicular planes.

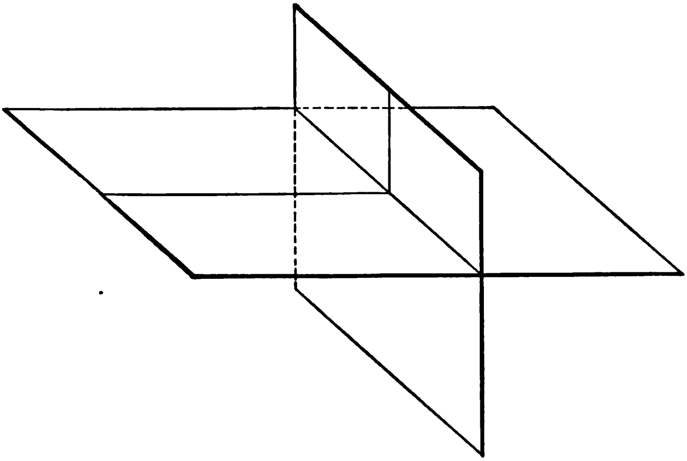


Fig. 11. Perpendicular planes showing a line in each, perpendicular to the line of intersection.

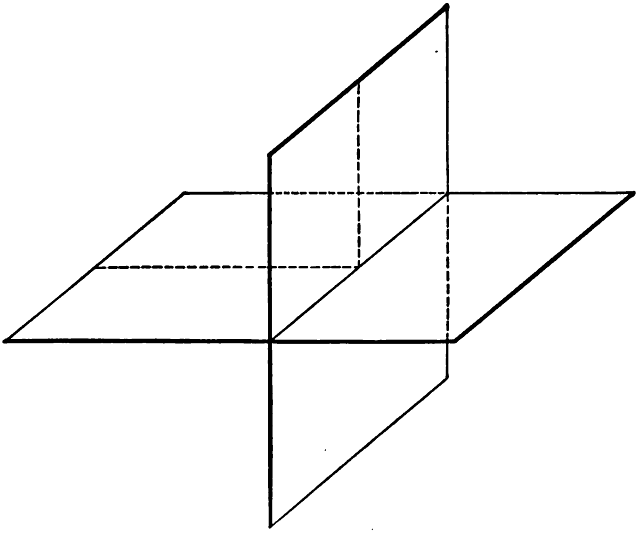


Fig. 12. Right view of perpendicular planes.

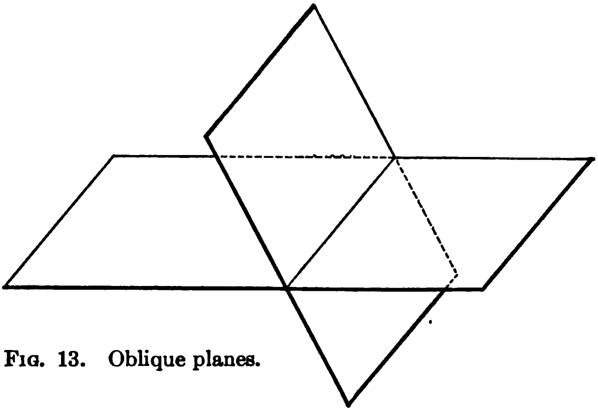


Fig. 13. Oblique planes.

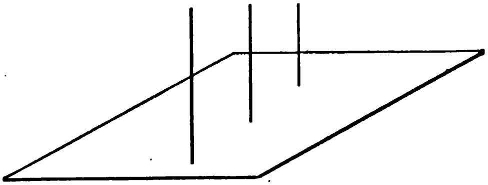


Fig. 14. Perpendiculars to a Plane.

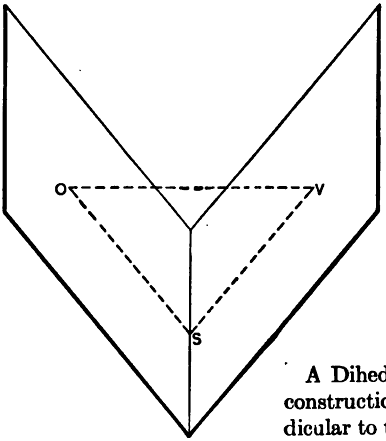


Fig. 15.
A Dihedral Angle with a construction plane perpendicular to the edge.

BOOK VIII

Lines and Planes

Dihedral Angles

Polyhedral Angles

1. Inters Pls St
2. ⊥ 2 Lines
3. At Pt 1 ⊥
4. Fr Pt 1 ⊥
5. ⊥ shortest
6. Thru Pt 1 ⊥ Pl
7. Line Inters ||
8. Pls ⊥ Line
9. All ⊥s Line
10. Line Inters ⊥
11. Lines ⊥ Pl
12. 1 || ⊥ Pl
13. Lines || Line
14. Dists =, Lines =
15. Dists ≠, Remote >
16. Lines =, Dists =
17. Greater Line Remote
18. Inters || Pls
19. ||s Bet || Pls
20. Line ⊥ 1 || Pls
21. Pls || same Pl
22. Internal Segs Pro
23. ∠s Diff Pls
24. Dihs =
25. Rat Dihs
26. Pls ⊥, Line in 1 ⊥ Inters
27. Pls ⊥, Line ⊥ 1 at Pt Inters
28. Pl Thru Line ⊥ Pl
29. Pls ⊥ Pl, Inters
30. 1Pl Thru Line non⊥ Pl
31. Pt in Bis Dih = ly
32. ∠ Line with Proj
33. Pl Thru 1 ||
34. Only 1 Common ⊥
35. Sum 2 Fc ∠s Trih
36. Sum Fc ∠s Polyhed
37. Trih ∠s =

VIII THEOREM I

The intersection of two planes is a straight line.[*]

Given planes I and II with the edges of I cutting II in points *A* and *B*.

Through *A* and *B* pass a straight line terminating in the edges.

A and *B* are in what plane?

Therefore all points on *AB* are in what plane?

(See Def. Plane.)

But *A* and *B* are also in what other plane?

Therefore all points on *AB* are in what other plane?

But how many points without *AB* are in I and II?

[See axiom 27 (2).]

Therefore *AB* is what part of I and II?

(See Def. 122.)

But *AB* is what kind of a line?

Therefore the intersection of I and II is what kind of a line?

VIII THEOREM 2

A line perpendicular to each of two other lines at their point of intersection, is perpendicular to their plane.

Draw a figure like the one on the next page, making the plane rather narrow.

Under it leave sufficient space to produce the perpendicular its length through the plane.

---

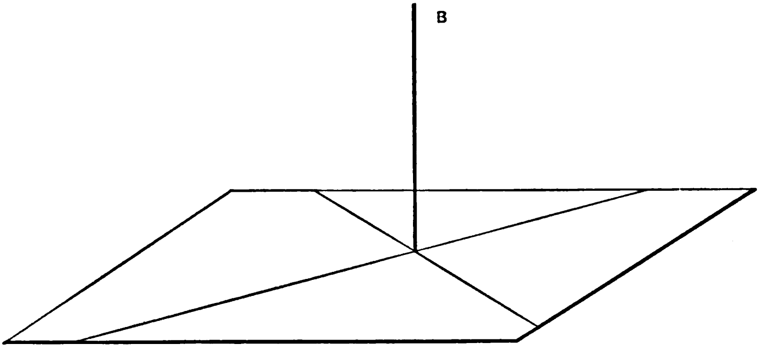
[*] In drawing all figures in solid geometry make rear lines light; face (front) lines heavy.

Use light construction lines to indicate edges of a plane concealed by another plane. (See page 152.)

A perpendicular to a plane is in what directional relation to all lines in the plane through the foot of the perpendicular?

It must therefore be proved that the given perpendicular is perpendicular to how many other lines in the plane in addition to the two named in the theorem?

In what way can all these other lines be represented in the figure?



Through *O* the foot of the perpendicular, draw in the plane such other line *OS* as may be required.

For a clear figure draw it nearly parallel to the right edge of the plane, terminating in the edges.

Extend the perpendicular its own length through the plane.

Draw a line *CD* cutting each of the three lines in the plane near the front edge.

Join both extremities of the produced perpendicular with each of the points of intersection of *CD* with the three lines in the plane.

In order to prove *BO* perpendicular to *OS* what relation must be shown between the angle *BOS* and its adjacent angle?

In order to show this relation, first prove the equality of the two triangles whose common side is the line cutting the lines in the plane, and whose vertices are at the extremities of the perpendicular as produced.

Make use of any necessary deduction from this equality in proving the equality of either of the two pairs of triangles whose common side is one of the segments of the line cutting the three lines in the plane, and whose vertices are at the extremities of the perpendicular as produced.

Finally prove the equality of the two triangles whose common side is the construction line *OS* or a segment of *OS*, and whose vertices are the extremities of the perpendicular as produced.

VIII THEOREM 3

At a point in a plane only one perpendicular to that plane can be drawn.

Given plane I and *OR* perpendicular to I at *R*.

Assuming it possible erect *any other* perpendicular to the plane at the same point, using a construction line.

How many planes can be passed through the perpendicular and the construction line?

Pass a plane* through these two lines and indicate its line of intersection with the given plane.

Finish the demonstration by considering the number of perpendiculars to the line of intersection.

VIII THEOREM 4

From a point without a plane only one perpendicular to that plane can be drawn.

Do not construct a second plane.

---

* A plane is said to be passed through a line when passed so that the plane contains the line; i.e., so that the line lies in the plane.

VIII THEOREM 5

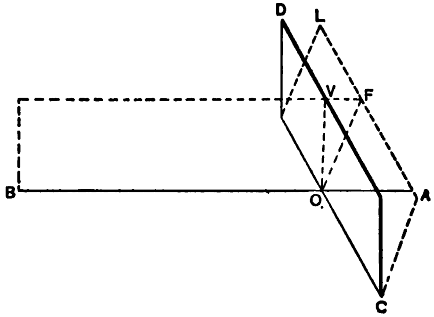
A perpendicular is the shortest line to a plane from a point without the plane.

Draw any other line from the point to the plane and join its extremity with the foot of the given perpendicular.

VIII THEOREM 6

Through a point in or without a line only one plane can be passed perpendicular to the line.

CASE I. Point *in* the line.



Given point O in the line AB and plane CD through O perpendicular to AB.

Assuming it possible draw CL any other plane through O perpendicular to AB.

Through AB pass a third plane BF cutting CD in OV and CL in OF.

CASE II. Point *without* the line.

Do not pass a third plane.

VIII THEOREM 7

If through a line which is parallel to a plane, a plane is passed cutting the given plane, the line of intersection with that plane is parallel to the given line.

VIII THEOREM 8

Planes perpendicular to the same line are parallel.

Pass a construction plane through the line and draw its line of intersection with each of the given planes.

SECOND DEMONSTRATION: Assume the planes *not* parallel.

VIII THEOREM 9

All perpendiculars to a line at a point in the line, lie in one plane, perpendicular to the line at the given point.

Given OR any perpendicular to AB at point O and plane LS perpendicular to AB at O.

Through AB and OR pass a plane and draw its line of intersection with LS.

VIII THEOREM 10

If from the foot of a perpendicular to a plane, a line is drawn in the plane at right angles to a second line in the plane, a line from their point of intersection to any point in the perpendicular is perpendicular to the second line in the plane.

From the point of intersection lay off equal distances on the second line in the plane.

Connect the extremities of the distances laid off with the foot of the perpendicular and with the given point in the perpendicular.

VIII THEOREM 11

Lines perpendicular to the same plane are parallel.

Connect the feet of the perpendiculars and the foot of one with the extremity of the other not in the plane.

Through the same foot draw a line in the plane, perpendicular to the joining line.

Therefore the lines are in how many planes?

Prove the other given perpendicular to be in the same plane.

VIII THEOREM 12

If one of two parallels is perpendicular to a plane, the other is perpendicular to the plane.

Assuming the other not to be perpendicular to the plane, draw a construction line perpendicular to the plane at the same point.

State the relation of this line to the first of the parallels and complete the demonstration, observing axiom 16.

VIII THEOREM 13

Two lines, each parallel to a third line, the three lines not being in the same plane, are parallel to each other.

Draw a construction plane, perpendicular to one of the three lines.

VIII THEOREM 14

Oblique lines from a point in a perpendicular to a plane, terminating in the plane at equal distances from the foot of the perpendicular, are equal.

Connect the foot of the perpendicular with the extremity of each oblique line.

VIII THEOREM 15

Oblique lines from a point in a perpendicular to a plane, terminating in the plane at unequal distances from the foot of the perpendicular, are unequal, and the one terminating at the greater distance from the foot of the perpendicular is the greater line.

Connect the foot of the perpendicular as in theorem 14.

In the same plane with either oblique line and the line joining the extremity of that oblique line to the foot of the perpendicular, draw a construction oblique line from the given point in the perpendicular, terminating in the plane at the same distance from the foot of the perpendicular as the other oblique line.

VIII THEOREM 16

Equal oblique lines drawn to a plane, from a point in a perpendicular to the plane, meet the plane at equal distances from the foot of the perpendicular.

VIII THEOREM 17

The greater of two unequal oblique lines drawn to a plane, from a point in a perpendicular to the plane, meets the plane at the greater distance from the foot of the perpendicular.

VIII THEOREM 18

The intersections of parallel planes by another plane are parallel.

VIII THEOREM 19

Parallel lines comprehended between parallel planes are equal.

Through the parallel lines pass a plane intersecting the given planes.

VIII THEOREM 20

A line perpendicular to one of two parallel planes is perpendicular to the other.

Through the line pass two planes intersecting the given planes, and draw the lines of intersection.

VIII THEOREM 21

Planes parallel to the same plane are parallel to each other.

Draw a line perpendicular to one of the planes.

VIII THEOREM 22

The corresponding internal segments of lines which are intersected by three or more parallel planes are proportional.

Draw three parallel planes.

In one of the outer planes mark two points a convenient distance apart, one near the front and the right edge, and the other near the rear and the left edge.

In the other outer plane mark two points near the lateral edges at equal distances from either the front or the rear edge.

Through the points pass the two given lines.

State the hypothesis and the conclusion.

Connect the point of intersection of one of the lines with one of the outer planes, to the point of intersection of the other line with the other outer plane.

Through the connecting line and one of the given lines, pass a plane represented by its lines of intersection with the middle plane and the outer plane in which the connecting line and the given line through which the plane is passed, have no common point.

Through the connecting line and the other given line, pass a plane represented by its lines of intersection with the middle plane and the outer plane in which no line of intersection has been drawn.

State the relation directionally of the last two lines of intersection, to each other.

In like manner state the relation to each other of the first two lines of intersection.

Refer to the algebraic expression of the conclusion and finish the demonstration.

VIII THEOREM 23

If angles in different planes have their sides respectively parallel and extending in the same direction from their vertices,

(1) *they are equal,*
(2) *their planes are parallel.*

For convenience draw two planes one above the other and draw the given angles.

From the vertices lay off equal distances on the respectively parallel sides.

In each plane draw a line connecting the extremities of the distances laid off.

Draw lines from one plane to the other, joining the vertices of all the angles each to each.

Prove the equality of the given angles by parallelograms.

Demonstrate the second part by the indirect method:

Assuming the planes not parallel, through the vertex of either given angle pass a plane parallel to the other given plane.

Consider planes to be passed through the line joining the vertices of the given angles and each of the lines joining the other vertices.

Draw the lines of intersection of each of these planes with the construction plane.

VIII Theorem 24

Dihedral angles are equal when their plane angles are equal.

Apply one dihedral angle to the other so that the plane angles coincide.

Prove the coincidence of the edges and the faces of the dihedrals, bearing in mind the theorem on the number of perpendiculars to a plane at a point in the plane, and the number of planes that may be passed through two intersecting lines.

VIII Theorem 25

Dihedral angles are in the same ratio as their plane angles.

Case I: **Plane angles commensurable.**

Divide the plane angles into a convenient number of equal parts and pass planes through the edge of each dihedral angle and the lines of division of its plane angle.

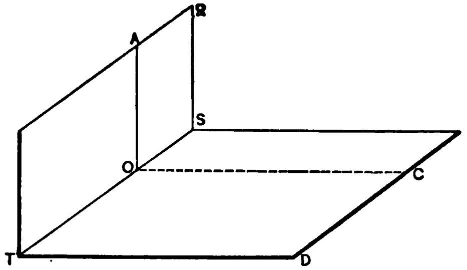
Case II: **Plane angles incommensurable.**

Take any convenient measure of one of the plane angles and apply it to the second plane angle as many times as containable.

Through the edge of the dihedral and the line marking the last division of the plane angle, pass a plane.

VIII Theorem 26

If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other plane.



VIII Theorem 27

If two planes are perpendicular to each other, a perpendicular to one of them at any point in their intersection lies in the other.

VIII THEOREM 28

If a line is perpendicular to a plane, every plane passed through the line is perpendicular to the given plane.

Pass any plane through the line and draw its line of intersection with the given plane.

VIII THEOREM 29

If each of two intersecting planes is perpendicular to a third plane, the line of intersection of the two is perpendicular to the third plane.

Assume the line of intersection not perpendicular, and draw a line perpendicular to the third plane at the common point of intersection of the two with the third.

Prove that this perpendicular lies in each of the other two, that no point without it is common to the two, and that therefore it is the line of intersection.

VIII THEOREM 30

Through a line not perpendicular to a plane, only one plane can be passed perpendicular to the given plane.

From any point in the line draw a perpendicular to the plane.

Through the line and the perpendicular pass a plane intersecting the given plane.

State the relation of the construction plane to the given plane.

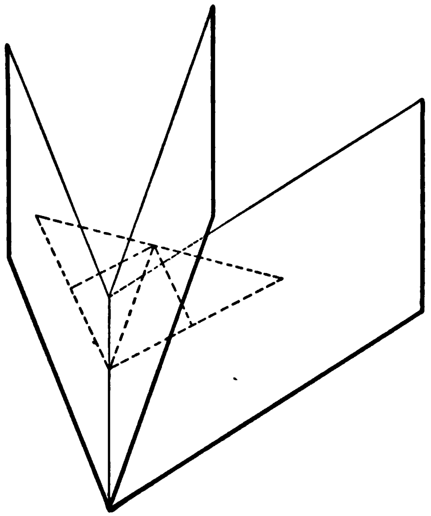
Assuming it possible, through the given line pass another plane perpendicular to the given plane.

State the relation of the given plane to the line of intersection of the two construction planes with each other.

VIII THEOREM 31

Every point in a plane which bisects a dihedral angle is equidistant from the faces of the angle.

From any point in the bisecting plane draw lines perpendicular to each face of the dihedral angle.



Through the perpendiculars pass a plane represented by its lines of intersection with the bisecting plane and with each of the faces of the given dihedral angle.

State the relation of the construction plane to each of the faces of the dihedral angle.

Therefore state the relation of the construction plane. to the edge of the given dihedral angle.

Finish the demonstration.

VIII THEOREM 32

The angle which a line makes with its projection on a plane is the least angle which it makes with any line in the plane.

Represent by a construction line a side of the angle which the line makes with any other line in the plane.

VIII THEOREM 33

A plane passed through one of two parallel lines and not the other, is parallel to the other.

In the plane draw any line parallel to the line through which the plane is passed.

Pass a plane perpendicular to one of the given lines or to the line drawn in the plane.

State the relation of the construction plane to each of the lines, and finish the demonstration.

VIII THEOREM 34

One line and one only, can be drawn as a common perpendicular to each of two other lines not in the same plane.

For convenience draw a figure in which the given lines are in the same relative position as shown on the next page.

Through *RS* pass a plane parallel to *OV*.

Draw its rear edge apparently parallel to *OV* and midway between *S* and *OV*, and its lateral edges apparently parallel to *RS*, one of them not less than an inch to the right of *RS*.

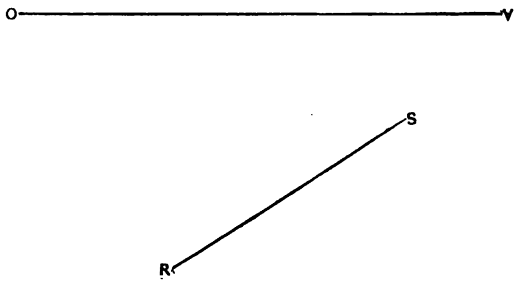
Through *OV* pass a plane perpendicular to the plane through *RS*.

Draw the line of intersection of the second plane with the first plane, representing it as cutting RS at some point between R and S.

In the plane passed through OV, at the point where the line of intersection cuts RS, draw a perpendicular to the line of intersection, terminating in OV.

Prove that this perpendicular is perpendicular to each of the given lines, bearing in mind the construction and two theorems previously demonstrated.

Prove that it is the *only* common perpendicular to the given lines by assuming it possible to draw any other common perpendicular LT to the lines, cutting them in any point except those in which they are cut by the line already proved to be a common perpendicular.



From the point of intersection of the assumed perpendicular with RS, draw in the plane containing RS, a line of convenient length parallel to the intersection of the two planes.

What is the relation of this line to OV?

Therefore what is the relation of the assumed perpendicular to the plane containing RS?

From the point of intersection of the assumed perpendicular with OV, draw a line perpendicular to the line of intersection of the two planes.

This line lies in what plane?

Therefore what is the line's relation to the plane containing RS?

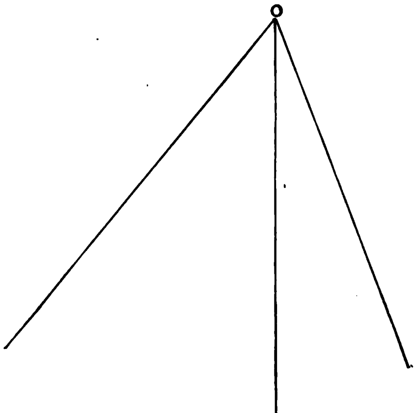
But what other line is in the same relation to that plane?

Therefore LT is *not* in what relation to OV and RS?

VIII THEOREM 35

The sum of any two face angles of a trihedral angle is greater than the third face angle.

Draw the figure as here represented, making the edge OV heavy. Denote right edge by OS.



For convenience let the rear face angle be greater than the front face angle VOS, and specify this relation in writing the hypothesis.

From the vertex draw in the rear face a line forming with the lateral edge OS, an angle equal to the front face angle VOS.

Through the three faces pass a plane cutting the front edge OV and the construction line in the rear face at equal distances from the vertex, representing this plane by drawing its lines of intersection with each face.

Make the lines of intersection with the two front faces, heavy.

State the relation of the sum of the two front face lines of intersection to the rear face line of intersection, expressing the rear face line of intersection in terms of its segments.

But one of the front face lines of intersection is equal to what segment of the rear face line of intersection?

Therefore the other segment of the rear face line of intersection is in what extensional relation to the other front face line of intersection?

What therefore is the relation of the angle opposite the last named front face line of intersection to the angle opposite the rear face line of intersection?

Finish the demonstration.

VIII THEOREM 36

The sum of the face angles of any convex polyhedral angle is less than four right angles.

Draw a convex polyhedral angle V of not less than four faces.

Through all of the faces pass a plane, outlined by its lines of intersection with the faces.

From any point O within the construction plane, draw lines to each of the vertices of the figure formed by the lines of intersection with the faces.

State the relation of the sum of the angles of the triangles whose vertices are at O (Sum $\angle$s $\triangle$s Verts O), to the sum of the angles of the triangles whose vertices are at the vertex of the given polyhedral angle (Sum $\angle$s $\triangle$s Verts at V).

But the sum of the angles at the bases of the triangles whose vertices are at O is in what relation to the sum of the angles at the bases of the triangles whose vertices are at the vertex of the given polyhedral angle?

Finish the demonstration.

VIII THEOREM 37

Trihedral angles whose face angles are respectively equal and whose parts are arranged in the same order, are equal.

Through the faces of each trihedral angle pass a plane which cuts all the edges at equal distances from the vertices and is represented by its lines of intersection with the faces.

On the edges of two correspondingly situated dihedral angles of the given trihedrals, mark off equal distances from the points of intersection of the construction planes with these edges.

At the extremities of the distances thus laid off, draw the plane angle of each dihedral, making the distances just named sufficiently short to have the sides of the plane angles terminate in the edges of the construction plane first passed through the faces of the trihedrals.

Join the extremities of the sides of the plane angles, in each construction plane.

In order that two dihedral angles may be proved equal, what must be proved to be the relation of their plane angles? *

Show that this relation exists between the two plane angles just drawn, and that therefore the dihedral angles are equal.

Superpose so that the equal dihedral angles and the vertices of the given trihedral angles coincide.

Finish the demonstration.

---

* Write answer at the bottom of the page of the work-book.

"The most difficult thing to get is intellectual sympathy and insight that will enable us to provide the conditions for another person's thinking, and yet allow him to do his own thinking in his own way."

JOHN DEWEY.

"If a man's wit be wandering, let him study the mathematics; for in demonstrations, if his wit be called away ever so little, he must begin again."

FRANCIS BACON.

"One achieves success by his parallelism to the course of thought which finds in him an obstructed channel."

EMERSON.

BOOK IX

POLYHEDRONS

PRISMS
PYRAMIDS

CYLINDERS

CONES

1. || Sects Prism =
2. Lat Area Prism
3. Prisms = when 3 Fcs

4. Opp Fcs || e π
5. Rat ▭ || e π = Bases
6. Rat ▭ || e π = Alts
7. Rat any ▭ || e π
8. Vol ▭ || e π
9. Vol Cube

10. Obl Prism ≎

11. Vol || e π
12. Plane Thru Edges || e π

13. Vol △ Prism
14. Vol Prism

15. Lat Area Reg Pyr
16. Lat Area Frus Reg Pyr
17. Sects || Base Pyr
18. Areas Sects || Base Pyr
19. Rat Sects || Base Pyrs = Alts

20. Sects Pyrs = Alts and ≎ Bases
21. Pyrs = Alts and ≎ Bases
22. Vol △ Pyr
23. Vol Pyr
24. Vol Frus △ Pyr
25. Vol Frus Pyr
26. Trunc △ Pyr ≎
27. Vol Trunc △ Pyr

28. Sect Cyl Thru Elem
29. ‖ Sects Cyl
30. Lat Area Cyl
31. Vol Cyl

32. Sect Cone Thru Vert
33. Sect Cone ‖ Base
34. Lat Area Cone Rev
35. Lat Area Frust Cone
36. Vol Cone
37. Vol Frust Cone

IX THEOREM 1

Sections of a prism made by parallel planes are equal polygons.

Draw a prism having not less than five lateral edges, representing the parallel planes by their lines of intersection with the faces.

IX THEOREM 2

The lateral area of a prism equals a lateral edge times the perimeter of a right section.

IX THEOREM 3

Prisms are equal if the three faces which include a trihedral angle of one are equal respectively to the three faces which include a correspondingly placed trihedral angle of the other.

Prove the equality of the trihedrals included by the respectively equal faces, and then superpose.

IX THEOREM 4

The opposite faces of a parallelepiped are equal and parallel.

IX THEOREM 5

Rectangular parallelepipeds having equal bases are in the same ratio as their altitudes.

CASE I: **Altitudes commensurable.**

Pass planes parallel to the respective bases, through the points of division of the altitudes by the common measure.

CASE II: **Altitudes incommensurable.**

Pass a plane parallel to the base, through the last point of division of the altitude and finish the demonstration.

IX THEOREM 6

Rectangular parallelepipeds having equal altitudes are in the same ratio as their respective bases.

Construct a third parallelepiped of the same altitude as the given parallelepipeds and whose base has one dimension of the base of one of the given parallelepipeds, and the other dimension equal to a dimension of the base of the other given parallelepiped.

IX THEOREM 7

Rectangular parallelepipeds are in the same ratio as the ratio of the products of their respective dimensions.

Draw two parallelepipeds which have no dimensions equal.

Construct a third parallelepiped whose altitude equals the altitude of one of the given parallelepipeds, and whose base equals the base of the other given parallelepiped.

IX THEOREM 8

The volume of a rectangular parallelepiped equals the product of its three dimensions.

Construct a unit volume.

IX THEOREM 9

The volume of a cube equals the cube of its edge.

IX THEOREM 10

An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude equals a lateral edge of the oblique prism.

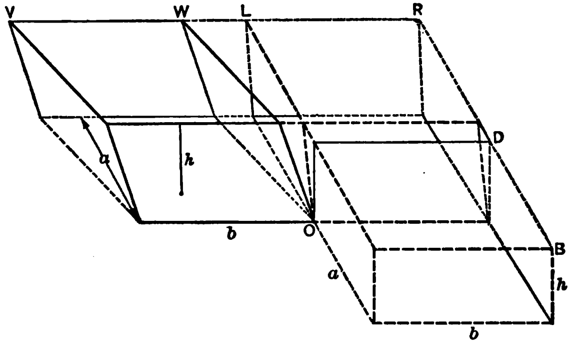
On the right section construct the right prism of the specified altitude, by producing the lateral edges of the oblique prism.

What kind of prisms are the two parts whose common base is the right section?

Prove their equality by the same method as that used in theorem 3 and finish the demonstration.

IX THEOREM 11

The volume of any parallelepiped equals the product of its base and altitude.



Draw an oblique parallelepiped OV whose bases are parallelograms. On a right section OL of the given parallelepiped, construct a right parallelepiped OR, making its altitude LR equal to a lateral edge VW of the given parallelepiped.

On a right section *OD* of this construction parallelepiped, construct a second right parallelepiped *OB* by the same method.

The bases of the second parallelepiped are what kind of figures?

The parallelepiped is therefore what kind of parallelepiped?

Finish the demonstration.

IX THEOREM 12

A plane through two diagonally opposite edges of a parallelepiped divides it into two equivalent triangular prisms.

IX THEOREM 13

The volume of a triangular prism, etc.

Write this theorem and demonstrate.

IX THEOREM 14

The volume of any prism, etc.

Write this theorem and demonstrate.

IX THEOREM 15

The lateral area of a regular pyramid, etc.

Write this theorem and demonstrate.

IX THEOREM 16

The lateral area of the frustum of a regular pyramid equals one-half the slant height times the sum of the perimeters of the bases.

IX THEOREM 17

If a pyramid be cut by a plane parallel to the base,
(1) *the edges and the altitude are divided proportionally,*
(2) *the section is a polygon similar to the base.*

To prove that the altitude is divided into segments proportional to the corresponding segments of the edges, pass a plane through any edge and the altitude, drawing its lines of intersection with the base of the pyramid and with the parallel section.

IX THEOREM 18

The areas of sections of a pyramid, parallel to the base, are proportional to the squares of their respective distances from the vertex of the pyramid.

IX THEOREM 19

If pyramids of equal altitude are each cut at the same distance from the vertex by a plane parallel to the base, the sections are in the same ratio as the respective bases.

IX THEOREM 20

If pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the respective bases at equal distances from their vertices, are equivalent.

IX THEOREM 21

Pyramids having equivalent bases and equal altitudes are equivalent.

Divide the altitude of each into any convenient number of equal parts, and through the points of division of both pass planes parallel to the bases, representing the planes by drawing their lines of intersection with the lateral faces of the pyramids.

On each of the sections thus formed in both pyramids, as an upper base, construct a prism whose lateral edges are parallel to any one of the lateral edges of the pyramid, and terminate in the lower section.

State the relation of the volume of each prism in one of the pyramids to the volume of the corresponding prisms in the other pyramid.

What therefore is the relation of the sum of the volumes of the prisms in one, to the sum of the volumes of the prisms in the other?

If the number of parts into which the altitude is divided is continually doubled, and prisms are continually formed as in the first division of the altitiude, the sums of the volumes of the successively resulting prisms become what kind of quantities?

But they are always in what relation?

Finish the demonstration.

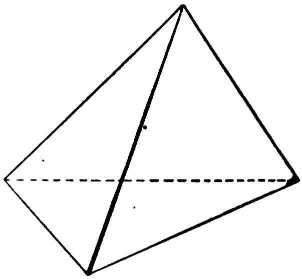
IX THEOREM 22

The volume of a triangular pyramid equals one-third the product of the base and the altitude.

Denote the volume by V and altitude by h.

On the base of the given pyramid construct a prism whose lateral edges are equal and parallel to one of the lateral edges of the pyramid.

Through one side of the upper base of the prism and one of the lateral edges of the pyramid, to which the lateral edges of the prism are *not* parallel, consider a plane to be passed and draw its line of intersection with the face of the prism.



State the relation of the two triangular pyramids whose bases are in the face in which the line of intersection was drawn, and whose vertices are in the same point as the vertex of the given pyramid.

State the relation of the given pyramid to the pyramid whose base is the upper base of the prism.

Compare the two expressions you have just written, and by this means determine to what part of the prism the given pyramid is equal.

IX THEOREM 23

The volume of any pyramid equals one-third the product of its base and altitude.

IX THEOREM 24

The volume of the frustum of a triangular pyramid equals one-third the altitude times the sum of the areas of its bases and the square root of the product of the areas of its bases.

Denote the volume by V, the altitude by h, and the areas of the bases by B and b respectively.

Pass planes through the frustum dividing it into three pyramids, two of which have as bases the bases of the frustum, representing the planes by their lines of intersection with the faces.

IX THEOREM 25

The volume of the frustum of any pyramid equals the sum of the areas of its bases and the square root of the product of the areas of its bases, multiplied by one-third the altitude of the frustum.

Produce the frustum so as to form a pyramid.

Construct a triangular pyramid whose base is equivalent to the base of the given frustum and whose altitude is equal to the altitude of the frustum as produced.

In what ratio are the volumes of the two pyramids?

Through the triangular pyramid pass a plane parallel to the base at the same distance from the vertex as the distance of the upper base of the frustum from the vertex, representing the plane by its lines of intersection with the faces.

Formulate the ratio of the volumes of the two pyramids whose bases are the upper bases of the frustum, and finish the demonstration.

IX THEOREM 26

A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism, and whose vertices are the three vertices of the inclined section (the section formed by the truncating plane).

Draw the figure as shown on the next page, so that no lateral edge is hidden by a face.

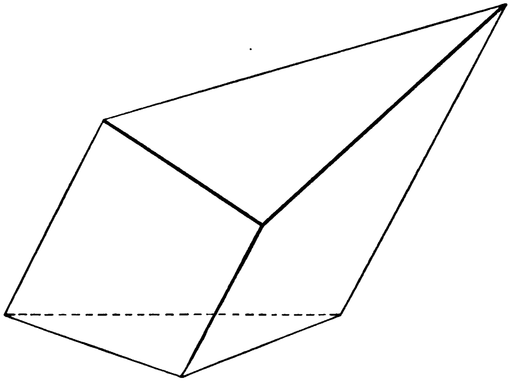
Through each edge of the lower base and the opposite vertex of the upper base pass planes, representing the planes by their lines of intersection with the faces.

Through the line of intersection with one of the front faces and the line of intersection with the rear face, pass a plane cutting off the portion of the prism having the longest lateral edge.

Now consider the given figure as cut by this plane and the plane previously passed through the rear edge of the lower base and the opposite vertex of the upper base.

Into how many parts is the figure divided by these two planes?

What is the shape of each of these parts?



Do any of the parts have a base and a vertex corresponding to those named in the conclusion?

That the theorem may be proved, to what must the other two parts be proved equal in volume?

In order to prove this, consider the two pyramids whose bases are in the rear faces, whose basal edge is the longest lateral edge of the given figure, and whose vertices are at opposite extremities of the front lateral edge.

Formulate the ratio of their volumes in terms of their respective bases and altitudes.

But their altitudes are in what relation?

Their bases if supposed to stand on the longer lateral edge of the given figure, have their areas in what relation?

The volumes of the two pyramids are therefore in what relation to each other?

Now consider the two intersecting pyramids having a common base in the rear lateral face and their vertices in the same points as the two pyramids already considered.

Proceed exactly as in the work just finished and complete the demonstration.

IX THEOREM 27

The volume of any truncated triangular prism equals the product of its right section by one-third the sum of its lateral edges.

Through each of the truncated prisms into which the right section divides the given truncated prism, pass planes as specified in the preceding demonstration, using the right section as a base for each part.

Formulate the volume of each of the two parts of the given figure by applying theorem 26, and complete the demonstration.

IX THEOREM 28

Every section of a cylinder formed by a plane passed through an element of the surface is a parallelogram.

IX THEOREM 29

Parallel sections cutting the elements of a cylinder are equal.

In any one of the sections inscribe a triangle.

Through each side of the triangle and an element of the surface pass planes, representing the planes by their lines of intersection with the cylinder and with the other section.

Show that circumferences of the sections coincide in three points when superposed, and therefore the sections are equal.

IX THEOREM 30

The lateral area of any cylinder equals the product of the circumference of a right section and an element of the surface.

Draw a cylinder whose elements are oblique to its bases and inscribe a prism.

IX THEOREM 31

The volume of a cylinder equals the product of its base and altitude.

Inscribe a prism.

IX THEOREM 32

Every section of a cone formed by a plane through its vertex is a triangle.

IX THEOREM 33

Every section of a cone formed by a plane parallel to the base is a circle.

In order to prove that the section is a circle what must be proved regarding every point in its boundary line?

Through the axis and any element of the cone pass a plane represented by its lines of intersection with the base, the parallel section, and the lateral face.

In like manner pass any other plane through the axis.

Prove the ratio of the lines of intersection of the two planes with the parallel section, equal to the ratio of their lines of intersection with the base, by showing similar triangles.

But what is the numerical value of the ratio of the two lines of intersection with the base?

Finish the demonstration.

IX THEOREM 34

The lateral area of a cone of revolution equals one-half the slant height times the circumference of the base.

Circumscribe a pyramid.

IX THEOREM 35

The lateral area of the frustum of a cone equals one-half the sum of the circumferences of the bases times the slant height.

IX THEOREM 36

The volume of any cone equals one-third the base times the altitude.

IX THEOREM 37

The volume of the frustum of a cone equals one-third the altitude times the continued sum of the areas of the bases and a mean proportional to (the square root of the product of) the areas of the bases.

BOOK X

THE SPHERE

1. Every section Sph
2. Line Jng Center Sph

3. ⊙s = Dist Fr Center
4. Nearer ⊙ >

5. Thru 2 Pts 1 ⌢
6. Thru 3 Pts

7. ∥ ⊙s
8. Great ⊙s =
9. Great ⊙ Bis Sph
10. Great ⊙s Bis
11. Great ⊙s Pls ⊥

12. Shortest Dist Bet 2 Pts
13. Pts Circum = Dist Fr Pole
14. Polar Dist Quad
15. Pt at Quad Dist

16. Pl ⊥ Rad
17. Pl Tan

18. Line in Tan Pl
19. Pl of 2 St Lines Ea Tan
20. Line Tan ⊙ of Sph

21. Sph Circum
22. Sph Inscr

23. 6 Pls Bis
24. Sph ∠ ▭

25. Ea Sd $<$
26. Sum Sds $< 360°$

27. 1 $\triangle$ Polar
28. Ea $\angle$ Polar $\triangle$ Supp
29. Sum $\angle$s $> 180°$ and $< 540°$

30. In Bi-rect Sph $\triangle$ Sds Quads
31. 3 Pls Ea $\perp$ other 2
32. Surface Sph $= 720°$
33. Sym Sph $\triangle$s

34. $\triangle$s 2 Sds and Incl $\angle$
35. $\triangle$s Sd and 2 Adj $\angle$s
36. Mut $= \angle$ lar $\triangle$s
37. Isos Sph $\triangle$
38. 2 $\angle$s $=$, $\triangle$ Isos
39. 2 $\angle$s $\neq$
40. 2 Sds $\neq$

41. Area Rev St Line

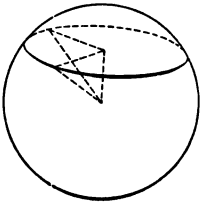
42. Surface Sph $= 4\pi R^2$
43. Surface Sph $= D \times 2\pi R$

44. Area Zone $= h \times 2\pi R$

45. Vol Sph $= \frac{1}{3} R \times A$
46. Vol Sph $= \frac{1}{6} \pi D^3$
47. Vols Sphs Pro Cubes
48. Vol Sph Sec
49. Vol Sph Seg
50. Vol Sph Seg 1 Base

X THEOREM 1

Every section of a sphere, formed by a plane, is a circle whose center is the projection of the center of the sphere, on the plane.



Locate any two points on the line of intersection of the surface of the sphere with the plane. To each of these points, draw radii of the sphere. Project the center of the sphere, on the plane. Connect the projection with each of the points marked on the line of intersection previously named.

X THEOREM 2

The line joining the center of a sphere to the center of a circle of the sphere is perpendicular to the plane of the circle.

X THEOREM 3

Circles of a sphere made by planes equally distant from the center, are equal.

X THEOREM 4

Of circles made by planes unequally distant from the center of a sphere, the one nearer to the center is the larger.

Draw a radius of each circle. Connect the center of the sphere with the extremity of each radius.

Prove by square of the hypotenuse.

X THEOREM 5

Through any two points on the surface of a sphere,

(1) *An arc of a great circle can be drawn.*

(2) *When the two points are not the extremities of a diameter of the sphere, only one can be drawn.*

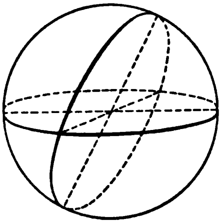
After demonstrating, show how many can be drawn when the two points *are* the extremities of a diameter.

X THEOREM 6

Through any three points on the surface of a sphere, one circle may be passed, and only one.

X THEOREM 7

Parallel circles have the same axis and the same poles.



X THEOREM 8

All great circles of a sphere are equal.

X THEOREM 9

Every great circle bisects the sphere.

X THEOREM 10

Great circles bisect each other.

X Theorem 11

Great circles whose planes are perpendicular pass through each other's poles.

X Theorem 12

The shortest distance on the surface of a sphere between any two points on that surface, is the arc (not greater than a semicircumference) of the great circle which joins them.

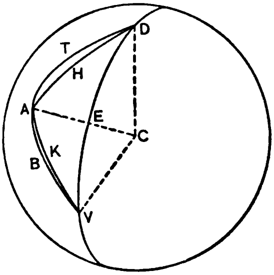
In the figure

D and *V* are the two given points;

DEV, the arc of a great circle;

DTBV, any line on the surface of the sphere, *except DEV*, joining *D* and *V*.

Through *D*, and any point *A* on *DTBV*, pass the arc of a great circle, *DHA*.



Through *V* and *A* pass the arc of a great circle, *AKV*.

Then *DHA*, *AKV*, and *VED* form what kind of a figure?

Therefore formulate the relation of *DEV* to *DHA*+*AKV*.

Continually double the number of points on *DTBV*, and through each and *V* and *D* assume arcs of great circles to be passed.

Is the summation of the arcs constant or variable?

What is its limit?

Finish the demonstration.

X Theorem 13

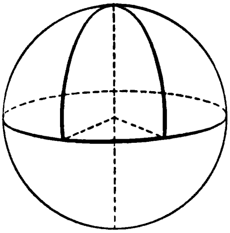
All points in the circumference of a circle of a sphere, are equally distant from the same pole of that circle.

X THEOREM 14

The polar distant of a great circle is a quadrant.

X THEOREM 15

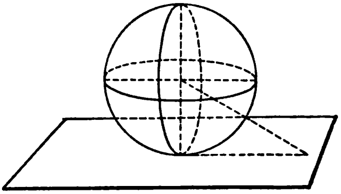
A point on the surface of a sphere, which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is the pole of the great circle passing through the two points.



Through the two points, and the center of the sphere pass a plane represented by its line of intersection with the surface of the sphere.

X THEOREM 16

A plane perpendicular to the radius of a sphere, at its extremity, is tangent to the sphere.



X THEOREM 17

Write this theorem as the converse of theorem 16.

X THEOREM 18

Any straight line in a tangent plane, through the point of tangency, is tangent to the sphere at that point.

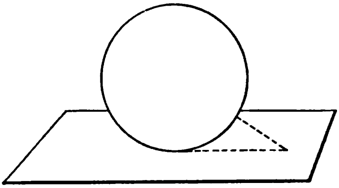
X THEOREM 19

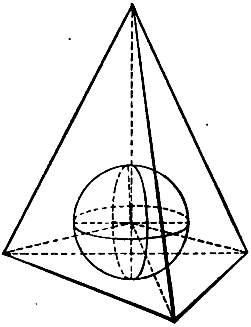
The plane of two straight lines each tangent to a sphere at the same point, is tangent to the sphere at that point.

X THEOREM 20

A straight line tangent to the circle of a sphere lies in a plane tangent to the sphere at the point of contact.

Assuming the line not to lie in the tangent plane, pass a plane through the line and the given plane, and draw the line of intersection with the given plane.





X THEOREM 21

Through four points not in the same plane, one spherical surface, only, may be passed, i.e., a sphere may be circumscribed about any tetrahedron.

X THEOREM 22

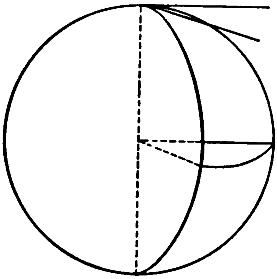
A sphere may be inscribed in any tetrahedron.

X THEOREM 23

The six planes which bisect the six dihedral angles of a tetrahedron intersect in the same point.

X THEOREM 24

A spherical angle is measured by the arc of a great circle included between its sides (produced if necessary), whose pole is the vertex of the angle.



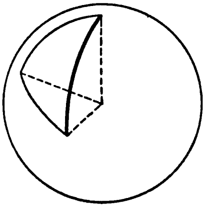
Draw the plane angle of the given spherical angle.

Draw radii of the great circle to its points of intersection with the sides of the given spherical angle, producing the sides if necessary.

Prove the angle formed by the two radii, equal to the plane angle.

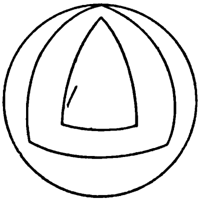
X THEOREM 25

Each side of a spherical triangle is less than the sum of the other two sides.



X THEOREM 26

The sum of the sides of a spherical polygon is less than 360°.



X THEOREM 27

If one triangle is the polar triangle of a second, the second is the polar triangle of the first.

X THEOREM 28

In polar triangles, each angle of the one triangle is the supplement of the side opposite to it in the other triangle.

Produce any two adjacent sides of the inner triangle until they meet a side of the outer triangle.

Formulate the number of degrees in the side of the outer triangle, included between its extremity and its point of intersection with the produced side of which the extremity is the pole; also the number of degrees between the other extremity of the same side and its point of intersection with the produced side of which that extremity is the pole.

X THEOREM 29

The sum of the angles of a spherical triangle is greater than 180° *and less than* 540°.

Draw the figure and construct its polar triangle.

Formulate the sum of each of its angles and the sides opposite in the construction triangle.

To what is the sum of its angles plus the sum of the sides of the construction polar triangle, therefore, equal?

Express in degrees the part of the first member of the equation, which is not required in the conclusion.

X THEOREM 30

In a bi-rectangular spherical triangle, the sides opposite the right angles are quadrants, and the side opposite the third angle measures that angle.

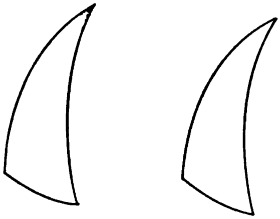
In what directional relation are the circles forming the right angles? Therefore where do they pass?

X Theorem 31

Three planes passed through the center of a sphere, each perpendicular to the other two planes, divide the surface of the sphere into eight tri-rectangular triangles.

X Theorem 32

The surface of a sphere equals 720 *spherical degrees.*

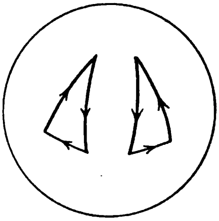


X Theorem 33

Symmetrical spherical triangles are equivalent.

X Theorem 34

Triangles on the same sphere, or equal spheres, are equal or equivalent, if two sides and the included angle of one are respectively equal to two sides and the included angle of the other.



X Theorem 35

Triangles on the same sphere, or equal spheres, are equal or equivalent, if a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.

X Theorem 36

Mutually equiangular triangles, on the same sphere, or equal spheres, are mutually equilateral, and are either equal or equivalent.

X Theorem 37

In an isosceles spherical triangle, the angles opposite the equal sides are equal.

X Theorem 38

Write this theorem as the converse of theorem 37.

X Theorem 39

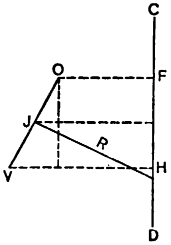
If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.

X Theorem 40

Write this theorem as the converse of theorem 39.

X Theorem 41

The area generated by the complete revolution of a straight line about a fixed straight line in the same plane equals the projection of the moving line on the fixed line times 2π *times a perpendicular bisector of the moving line, terminating in each line.*



In the figure OV is the revolving line; CD, the fixed line or axis;

$OVLD$, the area generated by the complete revolution of OV about CD;

FH, the projection of OV on CD;

R, a perpendicular bisector of OV.

Draw $OT \parallel CD$, terminating in VH.
Draw $JK \perp CD$, terminating in CH.

Now $OVLD$ is a frustrum of what?
$\therefore$ Area $OVLD$ equals JK times 2π times what?

But $\triangle OTV$ is in what relation to $\triangle JKH$?

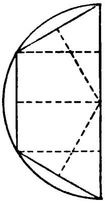
$$\therefore \frac{OV}{R}=\frac{T}{?}?$$

But OT = what? Substitute.

Take product of means equal to product of extremes and multiply the equation by 2π.

X THEOREM 42

The surface of a sphere equals four times the area of a great circle of the sphere.



In the great semicircle of the sphere inscribe half of a regular polygon.

From the center of the sphere draw a perpendicular to each side of the polygon.

Consider the half-polygon to make a complete revolution about the diameter of the sphere.

Then sum areas generated by sides of the half-polygon equals sum sides times what?

(Σ areas Gen Sds $= \Sigma$ Sds $\times$?).

Continually double the number of sides of semi-polygons, and continually assume a complete revolution.

Then Σ areas Gen by Sds becomes what?

And Σ Sds $\times$ what becomes what?

Finish the demonstration.

X THEOREM 43

The area of the surface of a sphere equals the diameter times the circumference of a great circle.

X THEOREM 44

The area of a zone equals its altitude times the circumference of a great circle.

X THEOREM 45

The volume of a sphere equals one-third the radius times the area of its surface.

Assume a polyhedron circumscribed about the sphere and a plane passed through the center of the sphere and each edge of the polyhedron.

Assume a radius to each face of the polyhedron. Then the planes cut the polyhedron into what kind of figures? Therefore sum of the volumes of the divisions of the polyhedron equals what?

(Σ Vols Poly = what?)

Continually double number of faces of the circumscribed polyhedron.

X THEOREM 46

The volume of a sphere equals one-sixth πD^3.

X THEOREM 47

The volumes of spheres are proportional to the cubes of their radii.

X THEOREM 48

The volume of a spherical sector equals one-third its radius times the area of its base.

X THEOREM 49

The volume of a spherical segment of two bases is expressed by the following formula:

$$V=\frac{\pi h}{2}(R^2+r^2)+\frac{\pi h^3}{6}$$

in which h = altitude of segment;
R = radius larger base;
and r = radius smaller base.

From A, the extremity of R, draw AC a radius of the sphere.

From B, the extremity of r, draw BC a radius of the sphere in the same plane as AC.

Denote the other extremity of R by T and the other extremity of r by V.

Assume $CBAT$ to make a complete revolution about a diameter through VT.

(Denote the extremities of this diameter by J and K, J being on the same side of the center as V.)

Then

(1) Vol Seg = Vol Sec ABC + Vol Cone ACV − Vol Cone BCT.

(2) But Vol ABC = Area Base $\times \frac{R}{3}$.

(3) But Base is a zone.

(4) ∴ Area Base = what?

(5) Substitute in (2) and simplify.

(6) Now Vol ACV = what?

(7) But AV or r is in what relation to JK?

(8) ∴ AV is in what relation to JV and VH?

(9) ∴ $\overline{AV}^2$ = what?

(10) But $JV = JC$ − what?

(11) And $VH = R_s$ + what?

(12) Substitute in (9) from (10) and (11).

(13) ∴ **Vol ACV** $= \frac{\pi}{3}(R^2 - \overline{VC}^2)VC$.

(14) Now Vol BCT = what?

Derive a value for R^2 exactly in the same way as in (7) to (12).

Substitute.

∴ **Vol BCT** $= \frac{\pi}{3}(R^2 - \overline{TC}^2)TC$.

By substitution in (1) obtain

$$\text{Vol Seg}=\frac{\pi}{3}[2R^2h+(R^2-\overline{VC}^2)VC-(R^2-\overline{TC}^2)TC]$$

Show that the second member reduces to

$$\frac{\pi}{3}[2R^2h+R^2(VC-TC)-(\overline{VC}^3-\overline{TC}^3)]$$

But $VC-TC=$ what?

Substitute, factor $\overline{VC}^3-\overline{TC}^3$, and obtain

(*a*) $\mathbf{Vol\ Seg}=\dfrac{\pi\mathbf{h}}{3}[\mathbf{3R^2-(\overline{VC}^2+VC\cdot CT+\overline{TC}^2)}]$

But $\overline{VC}^2+VC\cdot CT+\overline{TC}^2$

$$=\frac{3\overline{VC}^2}{2}-\frac{\overline{VC}^2}{2}+\frac{2VC\cdot CT}{2}+\frac{3\overline{CT}^2}{2}-\frac{\overline{CT}^2}{2};$$

$$=\frac{3\overline{VC}^2}{2}+\frac{3\overline{CT}^2}{2}-\frac{1}{2}(\overline{VC}^2-2VC\cdot CT+\overline{CT}^2)$$

Show that $\dfrac{h^2}{2}=\frac{1}{2}(\overline{VC}^2-2VC\cdot CT+\overline{CT}^2)$.

Substitute and obtain,

$$\mathbf{Vol\ Seg}=\frac{\pi\mathbf{h}}{3}\left[\mathbf{3R^2-\tfrac{3}{2}(\overline{VC}^2+\overline{CT}^2)-\frac{h^2}{2}}\right]$$

But by square hypotenuse

$\overline{VC}^2=$ what? *

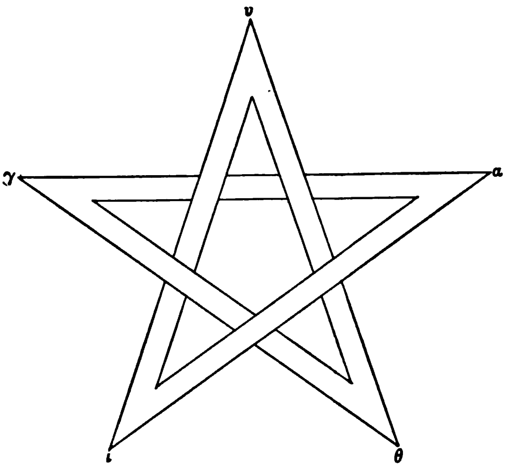
$\overline{CT}^2=$ what? *

Finish the demonstration.

X Theorem 50

The volume of a spheric segment of one base equals one-half π *times the altitude of the segment, times the sum of the square of the radius of the segment and one-third the cube of the altitude.*

* Denote radius of sphere by R_s.



Pentagram or five-pointed star, the mystic symbol of the Pythagoreans.

CONSTRUCTIONS

In the following constructions use only the straightedge and the compasses.

Draw as lightly as possible.

When arcs are described merely to determine their point of intersection, describe only about half an inch of each arc and ink in so that the intersection point lies apparently midway from the extremities.

Ink in, only after the entire construction has been made in pencil.

CONSTRUCTIONS

Problem 1. *To bisect a line.*

From each extremity as a center, with a radius greater than half the line, describe construction arcs, intersecting on both sides of the line. (Describe the arcs so that they will actually intersect, not simply meet.)

Prove that the line joining the two points of intersection bisects the given line.

Problem 2. *At a given point in a line, to erect a perpendicular to the line.*

First Method. Lay off equal distances on the line in each direction from the point, by using the point as a center and describing an arc cutting the line in two points. (If the point is at the extremity, produce the line.)

Finish the construction and prove.

Second Method. Locate a point without the line at any convenient distance from the given point. Using the point without as a center, with a radius equal to the distance of the point from the given point, describe an arc cutting the line in two points. (Produce the line if necessary.)

From the second point of intersection of the arc with the line, draw a line through the point without and produce it to meet the arc.

Join its meeting-point with the arc, to the given point in the line, and prove.

PROBLEM 3. *From a point without a line, to draw a perpendicular to the line.*

From the point as a center, with a radius longer than the distance from the point to the line, describe an arc cutting the line in two points.

Finish as in the preceding.

PROBLEM 4. *To bisect an arc.*

First Method. Describe arcs to intersect as in Problem 1, and prove.

Second Method. Draw the chord of the arc. Determine the center of the chord; through the center draw a perpendicular, and prove.

PROBLEM 5. *To bisect an angle.*

From the vertex as a center, with a convenient radius, describe an arc cutting the sides of the angle.

From each of the points of intersection as a center, with a radius somewhat greater than one-half the distance between the points of intersection, describe arcs intersecting each other without the first arc.

Finish and prove.

PROBLEM 6. *At a point in a line to construct an angle equal to a given angle.*

From the vertex of the angle as a center, with any convenient radius, describe an arc cutting the sides of the angle, and draw the chord of the arc intercepted by the angle.

From the given point in the line, as a center, with the same radius as before, describe an arc OV cutting the line.

From its point of intersection with the line, as a center, with a radius equal to the construction chord, describe an arc cutting OV.

Join the point of intersection to the given point in the line.

Prove.

Problem 7. *Through a point to draw a line parallel to a given line.*

Draw a line through the point to meet the given line at any angle.

Using the given point as the vertex and a segment of the construction line as a side, construct an angle equal to the angle formed with the given line.

Finish and prove.

Problem 8. *To divide a line into any number of equal parts.*

From one extremity of the line draw a line making an acute angle with the given line.

Lay off on the construction line the required number of equal parts, beginning at its point of intersection with the given line.

Through the last point of division of the construction line, draw a line OV to the extremity of the given line.

Through each of the other points of division draw a line parallel to OV, and prove.

Problem 9. *To divide a line into parts proportional to any number of given lines.*

Draw a given line OR nearly the width of the page.

For the given lines draw three short lines whose lengths are unequal.

From either extremity of the line which is to be divided, draw a construction line making a small acute angle with that line.

On the construction line, from its point of intersection V with the line to be divided, lay off each of the three given lines in sequence.

Connect the point V with the extremity of OR and through the other construction points draw parallels to the connecting line, cutting *OR*.

Prove.

PROBLEM 10. *To construct an equilateral triangle.*

PROBLEM 11. *To trisect a right angle.*

First Method. At a convenient distance from the vertex, locate a point *B* on one of the sides of the given angle. Denote the vertex by *A*. On *AB* construct an equilateral triangle.

Bisect its angle whose vertex is at *A*.

Prove.

Second Method. From the vertex of the angle, with a convenient radius *OV*, describe an arc cutting the sides.

From the points of intersection describe arcs with the same radius *OV*, cutting the first one.

Spanton's Approximate Method. From the vertex of the given angle, with a convenient radius describe an arc cutting the sides at *A* and *B*.

Draw the chord *AB*.

On *AB* as a diameter describe a semicircumference oppositely from the vertex.

With the same radius describe arcs from *A* and *B* cutting the semicircumference at *C* and *D*.

Join *V* with *T*, the middle point of the semicircumference, cutting arc *AB* in *H*.

Draw *RT*.

From *H* lay off *BT* on *VT* toward *V*.

Join the point thus determined to *C* and *D*, cutting arc *AR* in *E* and *S*.

Draw *VE* and *VS* in solid lines but lighter than the sides of the given angle.

Does the protractor show any error in the trisection?

Problem 12. *Given two angles of a triangle, to construct the third angle.*

Draw a line and at a point in the line lay off the two given angles.

Prove.

Problem 13. *Given two sides and the included angle of a triangle, to construct the triangle.*

Draw a line equal to either of the two sides, and at one extremity construct an angle equal to the given angle.

Finish the construction and prove.

Problem 14. *Given a side and two angles of a triangle to construct the triangle.*

Draw a line equal to the given side and at one extremity of the line construct an angle equal to one of the given angles; at the other extremity construct an angle equal to the other given angle.

Finish and prove.

(If one of the given angles is the angle opposite the given side, the third angle must be found as in Problem 8.)

Problem 15. *Given the three sides of a triangle, to construct the triangle.*

Draw a line equal to one of the sides.

From each extremity as a center with radii respectively equal to the other two sides, describe arcs which intersect.

Finish and prove.

Problem 16. *Given two sides of a triangle and the angle opposite one of them, to construct the triangle.*

Case I. Side opposite the given angle, less than the other given side.

Construct an angle equal to the given angle, and on one of the sides lay off from the vertex a distance equal to the longer of the two given sides of the triangle.

From the extremity of the distance laid off, with a radius equal to the other of the two given sides of the triangle, describe an arc, cutting the other side of the construction angle in as many points as possible.

Join the center of this arc to the cutting points, respectively.

Case II. Two given sides equal.

Case III. Side opposite the given angle, the greater.

Each of these cases should be worked with the given angle, respectively acute, right, and obtuse, when possible.

PROBLEM 17. *Given two sides and an included angle of a parallelogram, to construct the parallelogram.*

PROBLEM 18. *To describe a circumference through three points, not in a straight line.*

Join the first point to the second and the second to the third.

Draw perpendicular bisectors of each joining line.

Describe the circumference and prove.

PROBLEM 19. *To divide a circle into n equal parts.*

Divide the radius into n equal parts, say 5.

Or the radius as a diameter describe a semicircumference.

At each point of division of the radius, draw a perpendicular terminating in this semicircumference. Place one leg of the compasses at the center of the given circle, and with radii respectively equal to the distance from the center to the points of intersection of the perpendiculars with the semicircumference, describe concentric circles.

Prove.

PROBLEM 20. *To find the center of a circle or arc.*

PROBLEM 21. *To circumscribe a circle about a triangle.*

Bisect any two of the sides of the triangle, and at the points of bisection erect perpendiculars.

PROBLEM 22. *To inscribe a circle in a triangle.*

Draw the bisectors of two angles of the triangle.

PROBLEM 23. *Through a point to draw a tangent to a circle.*

Case I. Point on the circumference.

Case II. Point without the circumference.

In Case II connect the exterior point and the center of the circumference.

Bisect the connecting line; from its center, with a radius equal to one-half of the line, describe a circumference.

Join with the center of the given circle, the point of intersection of the construction circumference and the given circumference.

PROBLEM 24. *To find the geometric and the arithmetic mean between two given lines AR and RV.*

Geometric Mean. Lay off AR and RV oppositely from a point.

On AV as a diameter describe a semicircumference.

From R draw a perpendicular terminating in the semicircumference.

Prove that the perpendicular is the geometric mean between AR and RV.

Arithmetic Mean. By definition, the arithmetic mean between quantities equals the sum of the quantities divided by the number of quantities.

What line is the arithmetic mean between AR and RV?

PROBLEM 25. *To construct a square equivalent to the sum of two given squares.*

Lay off the sides of a right angle respectively equal to the sides of the given square, draw an hypotenuse, and finish the construction.

PROBLEM 26. *To construct a square equivalent to the difference of two squares.*

PROBLEM 27. *To draw a tangent to a circle at any point.*

Draw a radius to the point and a line perpendicular to the radius at its extremity.

PROBLEM 28. *To draw a common tangent to two equal circles.*

Draw the line of centers.

In each circle draw a radius perpendicular to the line of centers.

Join the extremities of the radii, and prove.

PROBLEM 29. *To draw common tangents to two unequal circles.*

Denote the given circles by I and II and their radii by R and r respectively.

From the center of the larger circle, with a radius equal to $R-r$, describe a circle III. Draw the line of centers and bisect it at M.

From M with a radius equal to half the line of centers, describe a circle cutting the circumference of III in A and B.

Join A and B to the center of III.

Draw radii of I and II perpendicular to both joining lines.

Join the extremities of the radii of I to the extremities of the radii of II, not crossing the line of centers.

Prove the lines, tangent.

PROBLEM 30. *To draw transverse * common tangents to two unequal circles.*

---

* Transverse means crossing the line of centers.

Use the same notation as in Problem 29.

From the center of I with a radius equal to $R+r$ describe a circle.

Make the same construction as in Problem 29 with the following exceptions:

(*a*) Use a radius equal to $R+r$.

(*b*) Join the extremities of the radii by transverse lines.

Prove the transverse lines, tangents.

PROBLEM 31. *At a point in a line without a circle, to draw a circle tangent to the circle and the line.*

Draw a perpendicular to the line at the given point.

Draw a produced diameter of the given circle, perpendicular to the given line produced if necessary.

Connect the unproduced extremity of the diameter with the given point.

From the center of the circle draw a line through the point of intersection of the connecting line with the circumference, and produce it to meet at O the perpendicular first drawn.

Prove that its point of intersection with that perpendicular is the center of the required circle, by proving a similarity of triangles, two of the sides of one of the triangles being radii of the given circle.

From O as a center describe the required circle.

PROBLEM 32. *To describe three equal circles tangent to each other.*

Draw a straight line OR equal to the given diameter. From each end of the line, with a radius equal to half the line, describe a circle.

On OR as one side construct an equilateral triangle.

From C the third vertex as a center, describe the third circle.

Prove.

PROBLEM 33. *To describe a circle tangent to three equal tangent circles.*

Draw the same figure as in Problem 32.

Bisect two of the angles of the equilateral triangle.

From the point of intersection of the biscetors describe the required circle.

Prove.

PROBLEM 34. *To describe equal circles around a circle of the same diameter, all tangent to each other and to the given circle.*

First Method. Make the same construction as in Problem 32, considering the circle whose center is *O* as the given circle. From the number of degrees in angle *O* determine how many circles may be drawn tangent to circle *O*, according to the conditions of the problem.

Describe the circles and prove.

Second Method. From the center of the given circle, with the diameter as a radius, describe a circle and divide its circumference into six equal parts by stepping off the radius as a chord as many times as possible.

From each point of division, with a radius equal to the radius of the given circle, describe a circle.

Prove.

PROBLEM 35. *Through points equally spaced along the side of a bent piece to be used as part of the felloe or rim of a wheel, to draw radial lines as a guide in boring the holes for the spokes.*

Clamp the piece with its curved edge against the edge of a planed board of the same thickness.

From the given points, with a radius greater than the distance between the points, describe arcs intersecting on the board.

Draw the radial lines across the rim.

Show the full construction in the work-book.

State how the lines might be drawn if the wheel were available.

PROBLEM 36. *To draw any regular polygon.*

Approximate Method of Antoine de Ville.

Draw a circle and its diameter.

Divide the diameter into as many equal parts as the number of sides of the required polygon, say nine.

From the ends of the diameter with a radius equal to the diameter, describe intersecting arcs.

Through the point of intersection of the arcs and the second * point of division of the diameter draw a straight line terminating in the circumference at *T* on the opposite side of the diameter. Take in the compasses the distance of *T* from the nearer end of the diameter, and step it off as a chord around the circumference.

By this method draw regular polygons of from five to fifteen sides inclusive, as assigned.

Second Approximate Method. Draw a straight line *TL* as one of the sides of the polygon.

From one end *T*, with the line as a radius, describe a semicircle, producing the line for its diameter.

With the protractor divide the semicircumference into the same number of equal parts as the number of sides of the required polygon.

Join *T* to *V*, the *second** point of division of the semicircumference, nearest *TL* as produced. Through *T*, *L*, and *V* pass a circumference.

Step off *TV* around the circumference and draw the polygon.

---

* Always the second, no matter how many sides.

As assigned draw regular polygons of from five to fifteen sides by this method.

Third Approximate Method. Draw *TL*, a side of the required polygon.

At *L* draw a perpendicular *LD* equal to *TL*.

From *L* as a center with a radius equal to *TL* describe an arc terminating in *T* and *D*.

Divide the arc *TD* into the same number of equal parts as the number of sides of the required polygon.

Join *L* to *S*, the second point of division nearest *D*.

Draw a perpendicular bisector of *TL* terminating in *LS* at *F*.

From *F* with *FL* as a radius, describe a circle. Step off *TL* around the circumference.

As assigned, draw polygons of from five to fifteen sides by this method.

PROBLEM 37. *To draw a regular pentagon.*

First Method. Draw *OV* as one side of the pentagon.

At *V* draw a perpendicular *VB* equal to *OV*.

From *E* the middle of *OV*, with *EB* as a radius, describe arc *BH* cutting at *H*, *OV* produced.

From *O* and *V*, with radius *OH* describe arcs intersecting at *A* on the same side of *OV* as *VB*. A is the third vertex of the pentagon.

Determine the other two vertices by describing intersecting arcs from *O*, *V*, and *A*, and draw the pentagon.

Second Method. Describe a circle and divide the circumference into five equal parts.

Through each of the five points of division draw a produced radius.

Draw a radius bisecting one of the five arcs, at *B*.

At *B* draw a tangent to the arc, cutting one of the produced radii at *C*.

Denote the center of the circle by *O*.

From *O*, with *OC* as a radius, describe short arcs cutting each of the produced radii. The points of intersection are the vertices of the required pentagon.

Draw the pentagon.

Third Method. Draw a side of the pentagon, *OV*.

Draw a perpendicular bisector of *OV* at *D*.

From *D* lay off on the bisector, *DR* equal to *OV*.

From *V* draw *VR* and produce it so that the produced part *RE* equals one-half *OV*.

From *V* as a center with a radius *VE*, describe a short arc cutting *DR*, produced if necessary, at *S*. *S* is the vertex of the pentagon opposite *OV*.

With radius *OV* describe short intersecting arcs from centers *S*, *O*, and *V*, and complete the pentagon.

Fourth Method. See Problem 7, pages 122–123.

Fifth Method. Draw a circle, its diameter, and a radius perpendicular to the diameter.

Join the middle point *F* of half the diameter, to the extremity of the radius.

With the joining line as a radius, from *F* as a center, describe a short arc cutting the diameter at *V*. The distance from *V* to the extremity of the radius which was drawn perpendicular to the diameter, is equal to a side of the required hexagon.

PROBLEM 38. *To draw a regular hexagon.*

Draw a circle, step off the radius on the circumference, and prove.

PROBLEM 39. *To draw a regular heptagon.*

First Method. Describe a circle and draw its radius *R*.

From the extremity of *R*, with *R* as a radius, describe an arc cutting the circumference at *A* and *B*. Draw the chord AB. Half of this chord equals the length of the side of the heptagon.

Second Method. Draw OV, a side of the heptagon.

From O, with radius OV describe a semicircle terminating at E in OV produced.

Draw a perpendicular bisector of VE, terminating in the semicircumference at J and in VE at K.

From E with radius KJ describe a short arc cutting the semicircumference at L.

Through O, V, and L pass a circumference.

Step off LV on the circumference.

Third Method. Draw a circle and its radius.

Draw a perpendicular bisector of the radius, terminating in the circumference and in the radius.

Step off the perpendicular bisector on the circumference.

PROBLEM 40. *To draw a regular octagon.*

PROBLEM 41. *To draw a regular nonagon.*

Draw a circle and two perpendicular diameters, TL and SC.

From the extremity T of one of the diameters, with a radius equal to the radius of the circle, describe a short arc cutting the circumference in H.

From L, the other extremity of the same diameter, with a radius LH, describe a short arc cutting at A the other diameter produced.

From A, with radius AT describe a short arc cutting SC at K.

Step off on the circumference the distance from K to the nearer extremity of SC.

PROBLEM 42. *To draw a regular decagon.*

Problem 43. *To draw a regular undecagon.*

Draw a circle and two perpendicular diameters, *TL* and *SC*.

From *T*, with a radius equal to the radius of the circle, describe a short arc cutting the circumference at *H*.

From the extremity of *SC* nearest *H*, with a radius equal to the radius of the circle, describe a short arc cutting the circumference at *B*, on the other side of *SC*.

From *H*, with radius *HB* describe a short arc cutting the diameter *TL* at *E*.

Step off the distance *BE* around the circumference.

Problem 44. *To draw a regular dodecagon.*

DEFINITIONS

PLANE AND SOLID GEOMETRY

This alphabetic list is inserted so that in case a particular definition is wanted, it may be immediately referred to.

In the study of the text, lists arranged in the order of the subject matter are of greater service. Such lists will be found on the following pages:

Definitions plane geometry, pp. 19–22.
Definitions solid geometry, pp. 137–148.

An Acute Angle is an angle less than a right angle.

Adjacent Angles are angles having a common side, a common vertex, and two sides exterior.

Adjacent Dihedral Angles are those having a common face and a common edge.

The Algebraic Square of a Line is the square of its numerical measure.

Alternate-Interior Angles are angles formed by two lines and a transversal, the angles being non-adjacent, within the two lines, and on opposite sides of the transversal.

Alternation is the interchange of the means or the extremes of a proportion.

The Altitude of a Cone is the perpendicular distance from the vertex to the base.

The Altitude of a Cylinder is the perpendicular distance between its bases.

The Altitude of a Frustum of a Cone is the perpendicular distance between its bases.

The Altitude of a Parallelogram is the perpendicular distance from the base to the opposite side.

The Altitude of a Prism is the perpendicular distance between the bases.

The Altitude of a Pyramid is a perpendicular distance from the apex to the base.

The Altitude of a Spherical Segment is the perpendicular distance between its bases.

The Altitude of a Triangle is a perpendicular from the vertex of any angle to the opposite side.

The Altitude of a Zone is the distance between the planes.

An Angle is a plane figure formed by two lines which meet at a point. The point is the **vertex** of the angle; the lines are the **sides** of the angle.

An Angle at the Center of a Regular Polygon is an angle formed by radii to the extremities of any side.

The Angle of a Lune is the angle between the semicircumferences which form its boundaries.

See definition, " The angle of two curves."

An Angle Measured by its Intercepted Arc is an angle whose number of angle degrees equals the number of intercepted arc degrees.

An Angle Measured by half into Intercepted Arc is an angle whose number of degrees equals one-half the number of intercepted arc degrees.

The Angle of Inclination of a Line to a Plane is the angle formed by the line with its projection on the plane.

The Angle of Two Curves Passing Through the Same Point is the angle formed by two straight lines tangent to he respective curves at the given point.

The Angles of a Spherical Polygon are the angles formed by the intersecting arcs.

The Antecedents in a proportion are the first terms of each ratio.

The Apothem of a Regular Polygon is the perpendicular distance from the center to any side.

An Arc is any part of the circumference of a circle.

The Area of a Polygon is its ratio to the unit of area; i.e., the ratio of its surface to the surface of the unit of area.

An Axiom is a self-evident, undemonstrable truth.

The Axis of a Circle of a Sphere is the diameter of the sphere which is perpendicular to the plane of the circle. The poles are the extremities of the axis.

The Axis of a Cone is a line from the vertex to the center of the base.

The Basal Edges of a Polyhedron are the lines in which the base intersects the faces.

The Base of a Polygon is the side upon which it stands or may be assumed to stand.

The Base of a Spherical Pyramid is a spherical polygon formed by the intersections of its sides with the surface of the sphere.

The Base of a Spherical Sector is the zone generated by the complete revolution of the arc of a circular sector about any diameter.

The Bases of a Spherical Segment are the sections made by the parallel planes which form it.

The Bases of a Zone are the circumferences of the sections made by the planes which form it.

A Bi-Rectangular Triangle is a spherical triangle having two right angles.

A Central Angle is an angle whose vertex is at the center of a circle and whose sides are radii.

The Center of a Regular Polygon is a point equally distant from the sides of a polygon.

A Circle is a plane figure from whose center all points of its boundary line are equidistant.

A Circular Cone is a cone whose base is a circle.

A Circular Cylinder is a cylinder whose bases are circles.

Elementary geometry treats of circular cylinders only. By cylinder, therefore, as used in geometry, a circular cylinder is understood.

The Circumference of a Circle is its boundary line.

A Circumscribed Polygon is a polygon whose sides are tangent to a circle. The circle is inscribed.

A Chord is a line whose extremities are in the circumference.

Coincident Figures are figures whose points are identical.

Commensurable Quantities are quantities which are evenly divisible by the same unit of measure.

Composition is the compounding of the terms of a proportion; i.e., the sum of the first and the second term, is to the first or the second, as the sum of the third and the fourth term, is to the third or the fourth.

Composition and Division is the compounding and the dividing of the terms of a proportion; i.e., the sum of the first and the second term, is to the difference of the first and the second term, as the sum of the third and the fourth term, is to the difference of the third and the fourth term.

Concentric Circles are circles having the same center.

A Conclusion is a statement of what is to be proved regarding the thing or things named in a hypothesis.

A Cone is a geometric solid whose lateral surface is conical and whose basal face is a plane cutting all the elements.

A Cone of Revolution is a cone generated by the revolution of a right triangle about one of the sides including the right angle.

A Conical Surface is a surface generated by the complete revolution of a line (called the generatrix), which passes through a fixed point (called the vertex), and constantly touches a fixed curve (called the directrix) not in the same plane with the generatrix.

The Consequents are the last terms of each ratio of a proportion.

A Constant is a quantity whose value does not change.

A Continued Proportion is a proportion of three or more ratios.

A Convex Polygon is a polygon no side of which, although produced, will enter the polygon.

A Convex Polyhedral Angle is a poly [illegible] le whose every section made by a plane cuttir [illegible] s, is a convex polygon.

A Convex Polyhedron is a polyhedron whose every section made by a plane, is a convex polygon.

A Convex Spherical Polygon is a spherical polygon whose corresponding polyhedral angle is convex.

In geometry, only convex spherical polygons are considered.

The Corresponding Polyhedral Angle of a Sphericla Polygon is the polyhedral angle formed at the center of the sphere by the great circles by whose arcs the spherical polygon is bounded.

A Corollary is a theorem easily deduced from another theorem.

A Cube is a rectangular parallelepiped whose dimensions are equal.

A Cylinder is a geometric solid whose lateral surface is cylindrical and whose two basal surfaces are plane and parallel.

A Cylinder of Revolution is a right circular cylinder. It may therefore be generated by the revolution of a rectangle about one side as an axis.

A Cylindrical Surface is a curved surface generated by a line called the generatrix, moving parallel to itself in continuous contact with, but not in the plane of, a curve called the directrix.

A Decagon is a polygon of ten sides.

A Degree is an angle which is one-ninetieth of a right angle. A right angle therefore, equals ninety degrees.

The degree is the unit of angular measure.

A Demonstration is a logically arranged series of statements, each based on mathematical authority, by means of which a theorem is proved.

A Diagonal of a Polyhedron is a line joining two vertices in different faces.

A Diagonal of a Quadrilateral is a line joining any two opposite vertices.

A Diagonal of a Spherical Polygon is an arc of a great circle connecting any two non-adjacent vertices.

A Diameter is a chord through the center of a circle.

A Diameter of a Sphere is a line through the center, whose extremities are in the surface.

A Dihedral Angle is a solid angle formed by two planes.

The Dimensions of a Rectangular Parallelepiped are three of its edges which meet at any vertex.

Division is the dividing of the terms of a proportion; i.e., the first minus the second term, is to the first or the second, as the third minus the fourth term, is to the third or the fourth.

A Dodecagon is a polygon of twelve sides.

A Dodecahedron is a polyhedron having twelve equal faces.

The Edge of a Dihedral Angle is the line in which its planes meet.

The Edges of a Polyhedral Angle are the lines in which the planes forming the polyhedral angle intersect.

The Edges of a Polyhedron are the lines in which the faces intersect.

An Element of a Conical Surface is a line representing the generatrix in any position.

The Element of Contact is the element of the surface contained by a tangent plane.

An Element of the Surface is a line representing the generatrix in any position.

An Endecagon is a polygon of eleven sides.

An Enneagon or Nonagon is a polygon of nine sides.

Equal Angles are angles which coincide or will coincide when in the same plane with one side and the vertex common, and one angle on the other.

Equal circles are circles whose radii are equal.

Equal Dihedral Angles are those which coincide or can be made to coincide.

Equal Figures are figures which coincide or can be superposed so as to coincide.

Equal Lines are lines of the same length, i.e., lines which are equal extensionally. They therefore coincide or can be made to coincide directionally and extensionally.

Equal Prisms are those which coincide or which can be made to coincide.

Equal Spherical Polygons are spherical polygons which coincide or can be made to coincide.

Equal Triangles are triangles whose homologous sides are equal and whose homologous angles are equal; they therefore coincide or can be made to coincide, i.e., they have the same size and shape.

Equimultiples of two quantities are products obtained by multiplying each of the quantities by the same quantity.

Equivalent Plane Figures are figures of the same size; i.e., figures of equal area.

Equivalent Solids are

(1) Solids of equal volume but of different shape:

(2) Solids having equal bases and altitudes respectively, but with parts arranged in a different order.

An Exterior Angle of a Triangle is an angle formed without the triangle, by any side and a produced side.

Exterior-Interior Angles are angles formed by two lines and a transversal, the angles being non-adjacent, on the same side of the transversal, one without the two lines, the other within.

The Exterior Sides of two adjacent angles are the outer sides, i.e., the sides which lie on opposite sides of the common side.

The Extremes are the first and fourth terms of a simple proportion.

A Face Angle of a Polyhedral Angle is an angle formed by two intersecting edges.

The Faces of a Dihedral Angle are the planes forming the angle.

The Faces of a Polyhedral Angle are the portions of the planes included between its edges.

The Faces of a Polyhedron are the planes which bound it, as limited by each other.

A Figure is a linear representation of the thing or relation of things, named in a theorem.

The Foot of a Perpendicular to a Plane is its point of intersection with the plane.

A Frustum of a Cone is the portion of the cone included between the base and the section formed by a plane cutting all its elements, parallel to the base.

A Frustum of a Pyramid is the portion of the pyramid remaining after cutting off the top by a plane parallel to the base.

A Great Circle of a Sphere is a section made by a plane which passes through the center of the sphere.

A Geometric Solid is a limited portion of unoccupied space of definite form, whose parts are not in one plane.

The Geometric Square of a Line is the square described on the line as a side.

A Heptagon is a polygon of seven sides.

A Hexagon is a polygon of six sides.

A Hexahedron is a polyhedron having six equal faces.

Homologous Angles are angles opposite equal sides in equal triangles, or opposite corresponding sides in similar triangles.

Homologous Sides are sides opposite equal angles in equal or similar triangles.

A Hypothesis is a statement of what thing or relation of things, is given in a theorem.

An Icosahedron is a polyhedron having twenty equal faces.

The Inclination of a Line to a Plane is the angle formed by the line with its projection on the plane.

Intersecting Lines are lines which cross each other.

The Intersection of Two Planes is the line containing all the points common to the planes, but no other points.

Incommensurable Quantities are quantities which have no common measure and are therefore not evenly divisible by the same quantity.

An Inscribed Angle is an angle whose vertex is in the circumference and whose sides are chords of a circle.

An Inscribed Polygon is a polygon whose sides are chords of a circle. The circle is circumscribed.

Inversion is the interchange of the means and the extremes of a proportion, i.e., an inversion of the ratios.

An Isosceles Triangle is a triangle having two equal sides. The equal sides are called legs.

The Lateral Area of a Prism is the area of its lateral faces.

The Lateral Area of a Pyramid is the area of the lateral faces.

The Lateral Edges of a Pyramid are the lines of intersection of the lateral faces.

The Lateral Surface of the Frustum of a Cone is the conical surface of the cone.

A Limit (geometric) is
(1) A constant;
(2) The constant which a variable continually approaches; and
(3) The constant whose numerical difference from the variable may ultimately be made indefinitely near to zero.

A Line of Centers is a line joining the centers of circles.

A Line Divided in Extreme and Mean Ratio is a line divided into two unequal segments, the greater of which is a mean proportional to the whole line and the shorter segment.

A Line Oblique to a Plane is a line perpendicular to only one line in the plane, through its point of intersection with the plane.

A Line, Parallel to a Plane, is a line parallel to its projection on the plane. It therefore will not meet the plane even when produced.

A Line, Perpendicular to a Plane, is a line perpendicular to all lines in the plane drawn through its point of intersection with the plane.

A Line, Tangent to a Cone, is a line having only one point in the surface of the cone, even when produced.

A Line, Tangent to a Cylinder, is any line in a tangent plane, cutting the element of contact.

A Line, Tangent to a Sphere, is a line having all points except one, the point of tangency, without the surface, even when produced.

The Lower Base of the Frustum of a Cone is the base of the cone.

A Lune is a portion of the surface of a sphere bounded by two semi-circumferences of great circles.

A Mean Proportional is a quantity which forms the means of a proportion of three quantities, i.e., a quantity which is in the same ratio to one quantity that it is to another inversely.

The Means are the second and third terms, i.e., the intermediate terms of a proportion.

A Median of a Triangle is a line joining any vertex to the middle of the opposite side.

The Nappes of a Conical Surface are the portions of the surface on each side of the vertex.

Non-Parallel Lines are lines which meet if produced in the direction of their convergence.

The Number of Degrees in a Spherical Angle is the number of degrees in its plane angle.

An Oblique Cylinder is a cylinder whose elements are oblique to its bases.

An Oblique Line is a line which makes neither a right angle nor a straight angle with another line.

An Oblique Prism is a prism whose lateral edges are oblique to its bases.

An Obtuse Angle is an angle greater than a right angle and less than a straight angle.

An Octagon is a polygon of eight sides.

An Octrahedron is a polyhedron having eight equal faces.

Parallel Circles are circles whose planes are parallel.

Parallel Lines are lines in the same plane, which will not meet within a finite distance, even when produced.

Parallel Planes are planes which do not meet within a finite distance, even when produced.

A Parallelepiped is a prism whose base is a parallelogram.

A Parallelogram is a quadrilateral whose opposite sides are parallel.

The Parts of a Polyhedral Angle are its face angles and its dihedral angles.

A Pentagon is a polygon of five sides.

A Pentadecagon is a polygon of fifteen sides.

The Perimeter of a Figure is the length of its boundary line; i.e., the distance around it.

A Perpendicular is a line which forms a right angle with another line, i.e., an angle of 90 degrees.

A Plane is a surface containing every point on a straight line through any two of its points.

The Plane Angle of a Dihedral Angle is the angle formed by two lines, one in each of the planes forming the dihedral angle, each perpendicular to the edge at the same point.

The Plane Angle of a Spherical Angle is the angle formed by tangents to the sides of the spherical angle at its vertex.

A Plane Figure is a figure all of whose points are in one plane.

A Plane, Perpendicular to Another Plane, is a plane which forms a right dihedral angle with that plane.

A Plane, Tangent to a Cone, is a plane containing only one element of the surface.

A Plane, Tangent to a Cylinder, is a plane containing only one element of the surface.

A Plane, Tangent to a Sphere, is a plane, all of whose points except one, the point of tangency, are without the sphere even when the plane is produced.

The Polar Distance of a Circle is the distance of its nearest pole from any point on its circumference.

Polar Triangles are spherical triangles in which the vertices of either are the respective poles of the sides of the other.

A Polyhedral Angle is a solid angle formed by three or more planes which meet in a common point.

A Polygon is a plane figure bounded by three or more straight lines.

A Polyhedron is a solid figure bounded by planes.

A Prism is a polyhedron whose lateral faces are parallelograms, and whose other two faces, called bases, are parallel to each other.

A Prism Circumscribed about a Cylinder is a prism whose bases are circumscribed about the bases of the cylinder.

A Prism Inscribed in a Cylinder is a prism whose bases are inscribed in the bases of the cylinder.

The Product of Two Lines is the product of their numerical measures.

The Projection of a Line upon Another is the segment of the second line included between the feet of the perpendiculars to that line from the extremities of the given line.

The projection upon the other therefore, of one of two lines which meet but do not intersect, as line *a* upon line *b*, is the segment of *b* included between the common point of *b* and *a*, and the foot of the perpendicular to *b* from the other extremity of *a*.

The Projection of a Line on a Plane is that part of a line in the plane, included between the feet of the perpendiculars to the plane from each extremity of the given line.

The Projection of a Point on a Plane is the foot of a perpendicular from the point to the plane.

A Proportion is an equality of ratios.

Proportional Quantities are quantities which form a proportion.

A Pyramid is a polyhedron whose base is a polygon and whose lateral faces are triangles having a common vertex called the apex of the pyramid.

A Pyramid Circumscribed about a Cone is a pyramid whose base is circumscribed about the base of the cone and whose apex coincides with the vertex of the cone.

A Pyramid Inscribed in a Cone is a pyramid whose base is inscribed in the base of the cone and whose apex coincides with the vertex of the cone.

A Quadrilateral is a polygon having four sides.

A Radius is a line from the center to the circumference of a circle.

The Radius of a Regular Polygon is the distance from the center of any vertex.

A Radius of a Sphere is a line from the center to the surface of the sphere.

The Ratio of Two Commensurable Quantities of the Same Kind is their relative magnitude expressed as a fraction, the numerator being the first quantity, the denominator the second quantity.

A Rectangle is a right-angled parallelogram.

A Rectangular Parallelepiped is a right parallelepiped whose bases are rectangles.

A Regular Polygon is a polygon which is equilateral and equiangular.

A Regular Pyramid is a pyramid whose base is a regular polygon in whose center will lie the projection of the apex of the pyramid.

A Right Angle is an angle formed by a vertical line meeting a horizontal line, or by a line making equal adjacent angles with another line.

A Right Circular Cone is a circular cone whose axis is perpendicular to its base.

A Right Cylinder is a cylinder whose elements are perpendicular to its bases.

A Right Dihedral Angle is a dihedral angle whose plane angle is a right angle.

A Right Parallelepiped is a parallelepiped whose lateral edges are perpendicular to its bases.

A Right Prism is a prism whose lateral edges are perpendicular to its bases.

A Right Section of a Cylinder is a section made by a plane perpendicular to its elements.

A Right Section of a Prism is a section formed by a plane perpendicular to the lateral edges.

A Right Triangle is a triangle having one right angle.

A Secant is a line having two points in the circumference and cutting the circumference in one or both of the points.

A Section of a Prism is a polygon formed by the intersection of a plane with the faces of the prism.

A Sector is a portion of a circle included by two radii and the intercepted arc.

A Segment is a portion of a circle included by a chord and its subtended arc.

The Sides of a Spherical Polygon are the arcs which bound it.

Similar Arcs are arcs of the same circle or of different circles, intercepted by equal angles at the center.

Similar Cones of Revolution are cones generated by the revolution of similar right triangles about the homologous sides including the right angles.

Similar Cylinders of Revolution are cylinders of revolution generated by similar rectangles revolving about homologous sides.

Similar Lunes are lunes of unequal spheres having equal angles.

Similar Polygons are polygons whose corresponding angles are equal and whose corresponding sides are proportional.

Similar Sectors are sectors of the same circle or of different circles, whose arcs are intercepted by equal angles at the center.

Similar Segments are segments of the same circle or different circles, whose arcs intercept equal angles at the center.

Similar Triangles are triangles whose corresponding angles are equal and whose homologous sides are proportional.

The Slant Height of a Cone is a line from the vertex perpendicular to a tangent to the base at any point.

The Slant Height of the Frustum of a Cone of Revolution is the element of contact of a plane, tangent to the frustum.

The Slant Height of a Regular Pyramid is the altitude of a lateral face.

A Small Circle of a Sphere is a section made by a plane which does not pass through the center of the sphere.

A Solid Angle is a solid figure formed by two or more planes which meet.

A Solid Figure is a figure whose parts are in more than one plane.

A Sphere is a geometric solid bounded by a surface, all of whose points are equally distant from the center.

A Sphere Circumscribed about a Polyhedron is a sphere in whose surface lie the vertices of the polyhedron.

A Sphere in a Polyhedron is a sphere to which each face of the polyhedron is tangent.

A Sphere of Revolution is a sphere generated by one complete revolution of a semicircle about its diameter.

A Spherical Angle is a figure formed by the intersection of the arcs of two great circles of a sphere.

A Spherical Degree is a bi-rectangular triangle which equals one-ninetieth of a tri-rectangular triangle.

The Spherical Excess of a Spherical Triangle is the difference between the sum of its angles and 180 spherical degrees.

A Spherical Polygon is the portion of the surface of a sphere bounded by three or more arcs of great circles.

A Spherical Pyramid is the portion of a sphere bounded by a spherical polygon and the planes of its sides.

A Spherical Sector is the portion of the sphere generated by the revolution of a circular sector about any diameter of the circle of which the sector is a part.

A Spherical Segment is a portion of a sphere contained between two parallel planes.

A Spherical Segment of One Base is a segment, one of whose planes is tangent to the sphere.

Unless otherwise specified, "spherical segment" signifies a segment of one base.

If a circular segment make a complete revolution about a diameter perpendicular to its chord, it generates a spheric segment of one base.

A Spherical Triangle is a spherical polygon of three sides; like a plane triangle, it may be right, oblique, equilateral, isosceles, or scalene.

A Spherical Wedge is a portion of a sphere bounded by a lune and two great semicircles.

A Square is an equilateral rectangle.

A Straight Angle is an angle whose sides are in the same straight line with its vertex, but in opposite directions.

A Straight * Line is a line which is completely determined by any two of its points.

Supplementary Angles are angles whose sum equals 180°. The supplement of an angle is therefore the difference between 180° and the angle.

Superposition is the movement of any part of a figure, or the placing of one figure in coincidence with any part of another in its plane.

Tangent Circles are circles whose circumferences have only one point in common.

A Tangent to a Circle is a line, all of whose points except one, the point of tangency, are without the circumference even when the line is produced.

Tangent Spheres are spheres whose surfaces have only one point in common.

The terms of a Proportion are the quantities forming the proportion.

A Tetrahedron is a polyhedron having four equal faces.

A Theorem is a truth admitting of, or requiring, demonstration.

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* Unless otherwise specified, line means straight line.

A Trapezoid is a quadrilateral two of whose sides are parallel and whose other two sides are not parallel.

A Triangle is a plane figure bounded by three straight lines.

A Triangular Prism is a prism whose base is a triangle.

A Trihedral Angle is a polyhedral angle having three faces.

A Tri-Rectangular Triangle is a spherical triangle having three right angles.

A Truncated Prism is either portion into which the prism is divided by a plane not parallel to its base.

A Unit of Area is a square whose side is one linear unit.

A Unit Circle is a circle whose radius is unity.

The Upper Base of the Frustum of a Cone is the section formed by the parallel plane.

A Variable is a quantity whose value changes continually.

The Vertex of a Polyhedral Angle is the point in which the planes meet.

The Vertex of a Spherical Pyramid is the point in which its faces meet at the center of the sphere.

Vertical Angles are the pairs of opposite angles formed by two intersecting lines.

The Vertices of a Polyhedron are the points in which the edges intersect.

The Vertices of a Spherical Polygon are the vertices of its angles.

The Vertex of a Triangle is the vertex of the angle opposite the base.

The Volume of a Rectangular Parallelepiped is its ratio to a unit rectangular parallelepiped (a unit of volume), each of whose dimensions is one linear unit.

A Zone is a portion of the surface of a sphere included between two parallel planes which cut the sphere.

A Zone of One Base is a zone having one of its bounding planes tangent to the sphere.

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